

The Input and Output Entropies of the k -Deletion/Insertion Channel with Small Radii

Shubhransh Singhvi*, Omer Sabary[‡], Daniella Bar-Lev[‡] and Eitan Yaakobi[‡]

*Signal Processing & Communications Research Center, International Institute of Information Technology, Hyderabad, India

[‡]Department of Computer Science, Technion—Israel Institute of Technology, Haifa 3200003, Israel

Abstract—The channel output entropy of a transmitted word is the entropy of the possible channel outputs and similarly the input entropy of a received word is the entropy of all possible transmitted words. The goal of this work is to study these entropy values for the k -deletion, k -insertion channel, where exactly k symbols are deleted, inserted in the transmitted word, respectively. If all possible words are transmitted with the same probability then studying the input and output entropies is equivalent. For both the 1-insertion and 1-deletion channels, it is proved that among all words with a fixed number of runs, the input entropy is minimized for words with a skewed distribution of their run lengths and it is maximized for words with a balanced distribution of their run lengths. Among our results, we establish a conjecture by Atashpendar et al. which claims that for the binary 1-deletion, the input entropy is maximized for the alternating words. For the 2-deletion channel, it is proved that constant words with a single run minimize the input entropy.

I. INTRODUCTION

In the last decade, channels that introduce insertion and deletion errors attracted significant attention due to their relevance to DNA storage systems [3], [19], [33], [34], [45], [49], where deletion and insertion are among the most dominant errors [23], [38]. The study of communication channels with insertion and deletion errors is also relevant to many other applications such as the synchronization of files and symbols of data streams [14] and for cases of over-sampling and under-sampling at the receiver side [39]. *VT codes*, designed by Varshamov and Tenengolts [46], are the first family of codes that correct single deletion or single insertion error [28]. Later, several works extended the scheme to correct multiple deletion errors, as well as substitutions; see e.g., [9], [16], [40], [41], [42]. Under some applications, such as DNA storage systems, the problem of list decoding was studied as well, for example in [2], [20], [22], [25], [31], [47]. Under this setup, the decoder receives a channel output and returns a (short) list of possible codewords that includes the transmitted codeword. Even though many works considered channels with deletion errors and designed codes that correct such errors, finding optimal codes for these channels and their capacity are yet far from being solved [11], [27], [32], [35]. Additionally, noisy channels that introduce deletion and insertion errors were also studied as part of the *trace reconstruction problem* [8] and the *sequence reconstruction problem* [29], [30]. In these problems, a codeword x is transmitted over the deletion channel multiple times. This transmission results in several noisy copies of x , and the goal is to find the required minimum number of these noisy copies that enables the reconstruction of x with high probability or in the worst case.

This work studies several related problem to the capacity of the deletion channel and the insertion channel and can contribute to their analysis. Assume a word x is transmitted over a communication channel and the channel output is y . This work first considers the *output entropy*, which is the entropy of the possible channel outputs y for a given input word x . Similarly, we study the *input entropy*, also studied in [4], which is the entropy of the possible transmitted words x , given a channel output y . Note that unlike the output entropy, the input entropy also depends on the probability to transmit each word x . Hence, it is assumed in this work that this probability is equal for each word x . Our main goal is to characterize these entropies, their expected values, and the sequences that maximize and minimize them for a combinatorial version of the insertion and the deletion channels, which are referred as the *k -deletion channel* and the *k -insertion channel*. In the k -deletion, k -insertion channel the number of deletions, insertions is exactly k and the errors are equally distributed [7], [18]. It was shown in [4] that the sequences that minimize the input entropy for the 1-deletion and the 2-deletion channels, in the binary case, are the all-zeroes and all-ones words, under the equal transmission probability assumption.

Studying the input and output entropies for the insertion and deletion channels can assist with improving existing results in various applications. The output entropy is directly related to the channel capacity [11], [15], [27], [32], [35], and can be used in list-decoders to rank the different codewords from the decoder's list by their transmission probability [20], [47]. The output entropy can also be used to assist with code design by prioritizing words with lower output entropy since they are likely to have higher successful decoding probability. Alternatively, when the goal is to encrypt the information, words with higher output entropies are preferred [5].

Among other applications, the input entropy is relevant to DNA reconstruction algorithms, since in several of these algorithms such as [6], [37], [44], there is a limitation on the number of noisy copies that are considered by the algorithm's decoder, due to design restrictions and run-time considerations. Therefore, in case the number of received noisy copies is greater than this limitation, a subset of these copies should be considered. Hence, to improve the accuracy of such algorithms, the input entropy of the channel outputs can be used to wisely select this subset of copies. Studying the input entropy was also investigated in [4], [5] for analyzing the information leakage of a key by revealing any of its subsequences.

The rest of the paper is organized as follows. In Section II we give the basic notations and definitions in the paper, discuss

the deletion channel capacity, and describe its relation to the input and output entropies of the k -deletion and k -insertion channels. Section II also gives a formal definition of the problems studied in this paper. Section III characterizes the input entropies of the k -deletion and k -insertion channels and Section IV finds their extremum values for $k = 1$ along with the minimum value for $k = 2$. Lastly, Section V studies the average input entropies for $k = 1$. Due to the lack of space, some of the proofs in the paper are omitted.

II. PRELIMINARIES AND PROBLEM STATEMENT

Let \mathcal{S} be a discrete channel with input alphabet \mathcal{X} and output alphabet \mathcal{Y} . The channel is characterized by a conditional probability distribution $\Pr_{\mathcal{S}}\{\mathbf{y} \text{ received}|\mathbf{x} \text{ transmitted}\}$, for every $\mathbf{x} \in \mathcal{X}^*$ and $\mathbf{y} \in \mathcal{Y}^*$, which is denoted for shorthand as $P_{\mathcal{S}}^{\text{Out}}(\mathbf{y}|\mathbf{x})$. The posterior probability of the channel \mathcal{S} is characterized by the conditional probability of the channel and the transmission probability of every message and is given by

$$\Pr_{\mathcal{S}}\{\mathbf{x} \text{ transmitted}|\mathbf{y} \text{ received}\} = \frac{P_{\mathcal{S}}^{\text{Out}}(\mathbf{y}|\mathbf{x}) \cdot P\{\mathbf{x} \text{ transmitted}\}}{P\{\mathbf{y} \text{ received}\}}.$$

We will refer to this probability in short as $P_{\mathcal{S}}^{\text{In}}(\mathbf{x}|\mathbf{y})$. When the channel will be clear from the context we may remove it from these notations. A channel \mathcal{S} is called a *discrete memoryless channel (DMC)* if it maps \mathcal{X}^n to \mathcal{Y}^n and

$$P_{\mathcal{S}}^{\text{Out}}(y_1, \dots, y_n | x_1, \dots, x_n) = \prod_{i=1}^n P_{\mathcal{S}}^{\text{Out}}(y_i | x_i).$$

It is well known that the channel capacity, i.e., the supremum of all achievable rates, of a DMC \mathcal{S} is given by

$$\text{Cap}(\mathcal{S}) = \max_{P(\mathbf{x})} I(X; Y), \quad (1)$$

where $P(\mathbf{x})$ is the transmission probability for every $\mathbf{x} \in \mathcal{X}^*$, X, Y are random variables denoting the transmitted and the received symbols over the channel, respectively, and $I(X; Y) = H(X) - H(X|Y) = H(Y) - H(Y|X)$ is the mutual information between X and Y .

Synchronization channels, such as the insertion channel and the deletion channel, are not memoryless and thus the connection in (1) does not necessarily hold for them. Hence, it is more common to study the capacity of the *finite block length*, which is denoted by $\text{Cap}_n(\mathcal{S})$ and is defined as

$$\text{Cap}_n(\mathcal{S}) = \frac{1}{n} \max_{P(\mathbf{x})} I(X^{(n)}; Y_{X^{(n)}}), \quad (2)$$

where the maximum is taken over all distributions $P(\mathbf{x})$ supported on \mathcal{X}^n and $Y_{X^{(n)}}$ is the random variable corresponding to the received message in this case. In general, it does not necessarily hold for synchronization channels that $\text{Cap}(\mathcal{S}) = \lim_{n \rightarrow \infty} \text{Cap}_n(\mathcal{S})$, however, based upon Dobrushin's result [15], several works explored conditions for such an equality to hold. Note that

$$I(X^{(n)}; Y_{X^{(n)}}) = H(Y_{X^{(n)}}) - \sum_{\mathbf{x} \in \mathcal{X}^n} \Pr(\mathbf{x}) H(Y_{X^{(n)}} | X^{(n)} = \mathbf{x}),$$

or, alternatively,

$$I(X^{(n)}; Y_{X^{(n)}}) = H(X^{(n)}) - \sum_{\mathbf{y} \in \mathcal{Y}^*} \Pr(\mathbf{y}) H(X^{(n)} | Y_{X^{(n)}} = \mathbf{y}).$$

Hence, one of the more important tasks in studying the capacity of synchronization channels is to determine for every $\mathbf{x} \in \mathcal{X}^n$ the conditional entropy $H(Y_{X^{(n)}} | X^{(n)} = \mathbf{x})$ and similarly for every $\mathbf{y} \in \mathcal{Y}^*$ the conditional entropy $H(X^{(n)} | Y_{X^{(n)}} = \mathbf{y})$. Formally, for an input $\mathbf{x} \in \mathcal{X}^n$, we refer to $H(Y_{X^{(n)}} | X^{(n)} = \mathbf{x})$ as the *output entropy* of the channel and for an output $\mathbf{y} \in \mathcal{Y}^*$, $H(X^{(n)} | Y_{X^{(n)}} = \mathbf{y})$ is the *input entropy* of the channel. Note that as opposed to several discrete symmetric memoryless channels, such as the binary symmetric channel and the binary erasure channel, the input and output entropies for synchronization channels, and in particular for the insertion and deletion channels, depend on the specific choice of \mathbf{x} and \mathbf{y} . Furthermore, while the output entropy depends solely on the channel \mathcal{S} , the input entropy depends both on \mathcal{S} and the channel input distribution $\Pr(X^{(n)})$. The case in which the channel input distribution is uniform is referred as *uniform transmission*.

This work studies the output and input entropies, while focusing primarily on the insertion channel and the deletion channel. Furthermore, to simplify the analysis of these problems, we consider the special case of the *k-deletion channel*, *k-insertion channel* which deletes, inserts exactly k symbols of the transmitted word uniformly at random, respectively.

For a positive integer n , let $[n] \triangleq \{1, \dots, n\}$ and let Σ_q be an alphabet of size q . For an integer $n \geq 0$, let Σ_q^n be the set of all sequences (words) of length n over the alphabet Σ_q . For an integer k , $0 \leq k \leq n$, a sequence $\mathbf{y} \in \Sigma_q^{n-k}$ is a *k-subsequence* of $\mathbf{x} \in \Sigma_q^n$ if \mathbf{y} can be obtained by deleting k symbols from \mathbf{x} . Similarly, a sequence $\mathbf{y} \in \Sigma_q^{n+k}$ is a *k-supersubsequence* of $\mathbf{x} \in \Sigma_q^n$ if \mathbf{x} is a k -subsequence of \mathbf{y} . Let \mathbf{x} and \mathbf{y} be two sequences of length n and m respectively such that $m < n$. The *embedding number* of \mathbf{y} in \mathbf{x} , denoted by $\omega_{\mathbf{y}}(\mathbf{x})$, is defined as the number of distinct occurrences of \mathbf{y} as a subsequence of \mathbf{x} . More formally, the embedding number is the number of distinct index sets, (i_1, \dots, i_m) , such that $1 \leq i_1 < \dots < i_m \leq n$ and $x_{i_1} = y_1, x_{i_2} = y_2, \dots, x_{i_m} = y_m$. For example, for $\mathbf{x} = 11220$ and $\mathbf{y} = 120$, it holds $\omega_{\mathbf{y}}(\mathbf{x}) = 4$. The *k-insertion ball* centred at $\mathbf{x} \in \Sigma_q^n$, denoted by $I_t(\mathbf{x}) \subseteq \Sigma_q^{n+t}$, is the set of all k -supersubsequences of \mathbf{x} . Similarly, the *k-deletion ball* centred at $\mathbf{x} \in \Sigma_q^n$, denoted by $D_t(\mathbf{x}) \subseteq \Sigma_q^{n-k}$, is the set of all k -subsequences of \mathbf{x} .

In the *k-deletion channel*, denoted by k -Del, exactly k symbols are deleted from the transmitted word. The k symbols are selected uniformly at random out of the $\binom{n}{k}$ symbol positions, where n is the length of the transmitted word. Hence, the conditional probability of k -Del is, $\Pr_{k\text{-Del}}^{\text{Out}}\{\mathbf{y}|\mathbf{x}\} = \frac{\omega_{\mathbf{y}}(\mathbf{x})}{\binom{n}{k}}$, for all $\mathbf{x} \in \Sigma_q^n, \mathbf{y} \in \Sigma_q^{n-k}$. Similarly, for the *k-insertion channel*, denoted by k -Ins, exactly k symbols are inserted to the transmitted word, while the locations and values of the k symbols are selected uniformly at random out of the $\binom{n+k}{k}$ possible locations and q^k possible options for the symbols. Therefore, the conditional probability of the k -insertion channel is, $\Pr_{k\text{-Ins}}^{\text{Out}}\{\mathbf{y}|\mathbf{x}\} = \frac{\omega_{\mathbf{x}}(\mathbf{y})}{\binom{n+k}{k} q^k}$, for all $\mathbf{x} \in \Sigma_q^n, \mathbf{y} \in \Sigma_q^{n+k}$.

Atashpendar et al. [4] studied the minimum input entropy of the k -Del, where $k \in \{1, 2\}$. In their work, they presented

an efficient algorithm to count the number of occurrences of a sequence \mathbf{y} in any given supersequence \mathbf{x} . They also provided an algorithm that receives a sequence \mathbf{y} , computes the set of all of its supersequences of specific length, characterizes the distribution of their embedding numbers and clusters them by their Hamming weight. Lastly, they proved that the all-zero and the all-one words minimize the input entropy of the 1-Del and 2-Del channels, under uniform transmission.

The goal of this work is to study the input entropy for the k -deletion and k -insertion channels as described below.

Problem 1. Find the following values for the channel k -Del:

- 1) For all $\mathbf{y} \in \Sigma_q^{n-k}$, find its input entropy over the channel k -Del with input distribution $\Pr(X^{(n)})$,

$$H_{k\text{-Del}, \Pr(X^{(n)})}^{\text{In}}(\mathbf{y}) \triangleq H(X^{(n)} | Y_{X^{(n)}} = \mathbf{y}).$$

- 2) Find the minimum, maximum, and average input entropy of the channel k -Del with input distribution $\Pr(X^{(n)})$,

$$\min_{k\text{-Del}}^{\text{In}}(n, \Pr(X^{(n)})) \triangleq \min_{\mathbf{y} \in \Sigma_q^{n-k}} \left\{ H_{k\text{-Del}, \Pr(X^{(n)})}^{\text{In}}(\mathbf{y}) \right\},$$

$$\max_{k\text{-Del}}^{\text{In}}(n, \Pr(X^{(n)})) \triangleq \max_{\mathbf{y} \in \Sigma_q^{n-k}} \left\{ H_{k\text{-Del}, \Pr(X^{(n)})}^{\text{In}}(\mathbf{y}) \right\},$$

$$\text{avg}_{k\text{-Del}}^{\text{In}}(n, \Pr(X^{(n)})) \triangleq \mathbb{E}_{\mathbf{y} \in \Sigma_q^{n-k}} \left\{ H_{k\text{-Del}, \Pr(X^{(n)})}^{\text{In}}(\mathbf{y}) \right\}.$$

The equivalent values and notations are defined similarly for the k -insertion channel, k -Ins.

Note that equivalent problems can be defined for the output entropy. In this work, unless otherwise stated, we assume a uniform transmission and we drop the $\Pr(X^{(n)})$ term from the notation of the studied entropy. In the following lemma we show the relation between the input entropy and the output entropy of the k -insertion and the k -deletion channels under uniform transmission.

Lemma 1. For uniform transmission over the k -insertion, k -deletion channel and for any channel output \mathbf{y} , it holds that,

$$H_{k\text{-Del}, \Pr(X^{(n)})}^{\text{In}}(\mathbf{y}) = H_{k\text{-Ins}}^{\text{Out}}(\mathbf{y}) \text{ and } H_{k\text{-Ins}, \Pr(X^{(n)})}^{\text{In}}(\mathbf{y}) = H_{k\text{-Del}}^{\text{Out}}(\mathbf{y}).$$

Hence, by Lemma 1, this work also solved the equivalent problems of the output entropy.

III. CHARACTERIZATION OF THE INPUT ENTROPY

This section presents a complete characterization of the input entropy of k -Del, k -Ins and gives an explicit expression of these entropies for any channel output \mathbf{y} .

The following definitions will be used in the rest of the paper. For a sequence \mathbf{x} , a *run* of \mathbf{x} is a maximal subsequence of identical consecutive symbols within \mathbf{x} . The number of runs in \mathbf{x} is denoted by $\rho(\mathbf{x})$. We denote by $\Sigma_{q,R}^n$, the set of sequences $\mathbf{x} \in \Sigma_q^n$, such that $\rho(\mathbf{x}) = R$. It is well known that the number of such sequences is, $|\Sigma_{q,R}^n| = \binom{n-1}{R-1} q(q-1)^{R-1}$.

For a sequence $\mathbf{x} \in \Sigma_{q,R}^n$, its *run length profile*, denoted by $\mathcal{RL}(\mathbf{x})$, is the vector of the lengths of the runs in \mathbf{x} . That is, $\mathcal{RL}(\mathbf{x}) \triangleq (r_1, r_2, \dots, r_R)$, where r_i , for $i \in [R]$ denotes the length of the i -th run of \mathbf{x} . For example, for $q = 4$ and $\mathbf{x} = 311221110$, we have that $\mathcal{RL}(\mathbf{x}) = (1, 2, 2, 3, 1)$. It

is said that $\mathbf{x} \in \Sigma_{q,R}^n$, is *skewed* if it consists of $R-1$ runs of length one, and a single run of length $n - (R-1)$; \mathbf{x} is *balanced* if \mathbf{x} consists of $r \equiv n \bmod R$ runs of length $\lceil \frac{n}{R} \rceil$ and the remaining $R-r$ runs are of length $\lfloor \frac{n}{R} \rfloor$. It is known that the size of k -insertion ball of any word $\mathbf{y} \in \Sigma_q^{n-k}$ is $|I_k(\mathbf{y})| = \sum_{0 \leq i \leq k} \binom{n}{i} (q-1)^i$ [28].

Lemma 2. For any integers n and k , such that $k \leq n$, and for any word $\mathbf{y} \in \Sigma_q^{n-k}$, we have that,

$$H_{k\text{-Del}}^{\text{In}}(\mathbf{y}) = \log \left(\binom{n}{k} q^k \right) - \frac{1}{\binom{n}{k} q^k} \sum_{\mathbf{x} \in I_k(\mathbf{y})} \omega_{\mathbf{y}}(\mathbf{x}) \cdot \log(\omega_{\mathbf{y}}(\mathbf{x})).$$

In the special case of $k = 1$, we have the next corollary.

Corollary 1. $H_{1\text{-Del}}^{\text{In}}(\mathbf{y})$ is invariant to permutations of run length profile of the channel output $\mathbf{y} \in \Sigma_q^{n-1}$ and it is given by the following expression

$$H_{1\text{-Del}}^{\text{In}}(\mathbf{y}) = \log(nq) - \frac{1}{nq} \sum_{i=1}^{\rho(\mathbf{y})} (r_i + 1) \log(r_i + 1),$$

where $r_i, i \in [\rho(\mathbf{y})]$, denotes the length of the i -th run of \mathbf{y} .

Proof. From [7], [36], $I_1(\mathbf{y})$ has $\rho(\mathbf{y})$ sequences that can be obtained by prolonging an existing run, each with embedding weight $r_i + 1$ for $1 \leq i \leq \rho(\mathbf{y})$. The embedding weight of the remaining $|I_1(\mathbf{y})| - \rho(\mathbf{y}) = q + |\mathbf{y}|(q-1) - \rho(\mathbf{y})$ sequences is one. Therefore from Lemma 2, it follows that,

$$H_{1\text{-Del}}^{\text{In}}(\mathbf{y}) = \log(nq) - \frac{1}{nq} \sum_{i=1}^{\rho(\mathbf{y})} (r_i + 1) \log(r_i + 1).$$

□

Similarly, we have the next lemma for the k -Ins channel.

Lemma 3. For any integers n and k , such that $k \leq n$, and for any word $\mathbf{y} \in \Sigma_q^{n+k}$, we have that,

$$H_{k\text{-Ins}}^{\text{In}}(\mathbf{y}) = \log \left(\binom{n+k}{k} \right) - \frac{1}{\binom{n+k}{k}} \sum_{\mathbf{x} \in D_k(\mathbf{y})} \omega_{\mathbf{y}}(\mathbf{x}) \cdot \log(\omega_{\mathbf{y}}(\mathbf{x})).$$

In particular, for $k = 1$,

$$H_{1\text{-Ins}}^{\text{In}}(\mathbf{y}) = \log(n+1) - \frac{1}{n+1} \sum_{i=1}^{\rho(\mathbf{y})} r_i \log(r_i),$$

where $r_i, i \in [\rho(\mathbf{y})]$ denotes the length of the i -th run of \mathbf{y} .

IV. EXTREMUM VALUES OF THE INPUT ENTROPY

In this section we first find the channel outputs that have maximum and minimum entropy among all sequences in $\Sigma_{q,R}^m$, where $R \in [m]$, for $m \in \{n-1, n+1\}$. Then as corollaries, we find the maximum and minimum values of the input entropies of the channels 1-Del and 1-Ins. We then derive the extremum values of input entropies for 2-Del and 2-Ins.

A. The Single-Deletion Channel: Maximum Input Entropy

This subsection studies the channel outputs $\mathbf{y} \in \Sigma_q^m$ that maximizes the input entropy of 1-Del, where $m \triangleq n-1$.

Lemma 4. Let $R \in [m]$ and $r \equiv m \bmod R$, the maximum input entropy among all channel outputs in $\Sigma_{q,R}^m$ is

$$\begin{aligned} \max_{\mathbf{y} \in \Sigma_{q,R}^m} H_{1\text{-Del}}^{\text{In}}(\mathbf{y}) &= \log(nq) - \frac{r}{nq} \left(\left\lceil \frac{m}{R} \right\rceil + 1 \right) \log \left(\left\lceil \frac{m}{R} \right\rceil + 1 \right) \\ &\quad - \frac{R-r}{nq} \left(\left\lfloor \frac{m}{R} \right\rfloor + 1 \right) \log \left(\left\lfloor \frac{m}{R} \right\rfloor + 1 \right), \end{aligned}$$

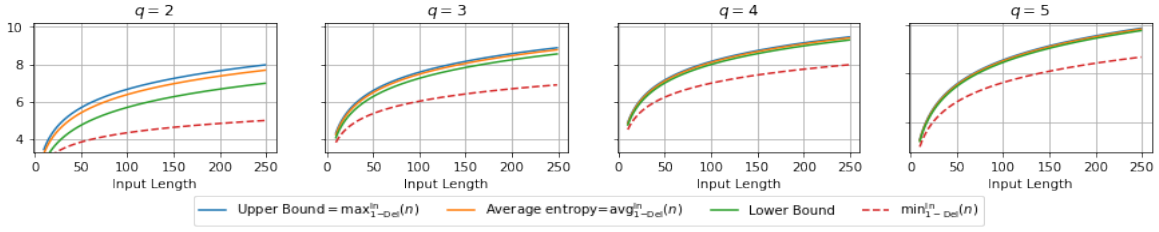


Fig. 1: The minimum, maximum, average values and a lower bound on the average of the input entropy of the 1-deletion channel.

and is only attained by balanced channel outputs.

Proof. Let $\mathbf{y} \in \Sigma_{q,R}^m$ be a sequence with maximum entropy and assume its run length profile is (r_1, r_2, \dots, r_R) . Assume to the contrary that \mathbf{y} is not one of the balanced sequences. Then, it can be verified that there exist indices $\ell \neq s$ such that $r_s \leq \lfloor \frac{m}{\rho(\mathbf{y})} \rfloor - 1$ and $\lfloor \frac{m}{\rho(\mathbf{y})} \rfloor < r_\ell$ or $r_s < \lfloor \frac{m}{\rho(\mathbf{y})} \rfloor - 1$ and $\lfloor \frac{m}{\rho(\mathbf{y})} \rfloor \leq r_\ell$ and in particular $r_\ell - r_s \geq 2$. Consider a sequence $\mathbf{y}' \in \Sigma_{q,R}^m$, with run length profile $(r'_1, r'_2, \dots, r'_R)$, where $r'_\ell = r_\ell - 1$, $r'_s = r_s + 1$, and for any $k \notin \{\ell, s\}$, $r'_k = r_k$. Now, consider the entropies difference

$$\begin{aligned} H_{1\text{-Del}}^n(\mathbf{y}') - H_{1\text{-Del}}^n(\mathbf{y}) &= -\frac{1}{nq} (r_\ell \log r_\ell + (r_s + 2) \log(r_s + 2) \\ &\quad - (r_\ell + 1) \log(r_\ell + 1) - (r_s + 1) \log(r_s + 1)). \end{aligned}$$

Let $g(r) \triangleq (r+1) \log(r+1) - r \log(r)$, and note that g is an increasing function w.r.t. r . In addition, since $r_\ell > r_s + 1$,

$$H_{1\text{-Del}}^n(\mathbf{y}') - H_{1\text{-Del}}^n(\mathbf{y}) = \frac{g(r_\ell) - g(r_s + 1)}{nq} > 0.$$

This is a contradiction to the assumption on \mathbf{y} . Therefore, among all sequences in $\Sigma_{q,R}^m$, the maximum entropy is attained by the balanced channel outputs. The value of this maximum entropy can be simply derived from the run length profile of the balanced channel outputs. \square

Corollary 2. The maximum input entropy among all channel outputs in Σ_q^m is

$$\max_{1\text{-Del}}^n(n) = \log(nq) - \frac{2m}{nq}$$

and is only attained by channel outputs with m runs.

Proof. Let $\mathbf{y} \in \Sigma_q^m$ and let $\mathcal{RL}(\mathbf{y}) = (r_1, r_2, \dots, r_{\rho(\mathbf{y})})$. From Corollary 1, it can be shown that,

$$H_{1\text{-Del}}^n(\mathbf{y}) = \log(nq) + \frac{m + \rho(\mathbf{y})}{nq} \sum_{i=1}^{\rho(\mathbf{y})} \frac{r_i + 1}{m + \rho(\mathbf{y})} \log\left(\frac{m + \rho(\mathbf{y})}{r_i + 1}\right)$$

$$\text{Since } \sum_{i=1}^{\rho(\mathbf{y})} \frac{r_i + 1}{m + \rho(\mathbf{y})} = 1, \text{ Jensen's inequality implies that}$$

$$H_{1\text{-Del}}^n(\mathbf{y}) \leq \log(nq) + \frac{m + \rho(\mathbf{y})}{nq} \log\left(\frac{\rho(\mathbf{y})}{m + \rho(\mathbf{y})}\right).$$

Let $f(x) : [1, m] \rightarrow \mathbb{R}$ be defined as

$$f(x) \triangleq \log(nq) + \frac{m + x}{nq} \log\left(\frac{x}{m + x}\right).$$

f is increasing w.r.t x and hence, for $r \in [1, m]$, $f(r) \leq f(m)$, and equality is attained if and only if $r = m$. Hence, among all sequences in Σ_q^m , channel outputs with $\rho(\mathbf{y}) = m$ have the maximum entropy $\max_{1\text{-Del}}^n(n) = \log(nq) - \frac{2m}{nq}$. \square

B. The Single-Deletion Channel: Minimum Input Entropy

Similarly to the previous subsection, here we study the channel outputs $\mathbf{y} \in \Sigma_q^m$ that minimize the input entropy.

Lemma 5. Let $R \in [m]$, the minimum input entropy among all channel outputs in $\Sigma_{q,R}^m$ is

$$\min_{\mathbf{y} \in \Sigma_{q,R}^m} H_{1\text{-Del}}^n(\mathbf{y}) = \log(nq) - \frac{(m-R+2) \log(m-R+2) + 2(R-1)}{nq}$$

and is attained only by skewed channel outputs.

It is easy to verify that $\min_{\mathbf{y} \in \Sigma_{q,R}^m} H_{1\text{-Del}}^n(\mathbf{y})$ increases with R and hence we have the following corollary which states that the channel outputs that minimize this entropy have a single run. This extends the results from [4] to the non-binary case.

Corollary 3. The minimum input entropy among all channel outputs in Σ_q^m is $\min_{1\text{-Del}}^n(n) = \log(nq) - \frac{\log(n)}{q}$ and is only attained by channel outputs having a single run.

C. The Single-Insertion Channel

Using similar techniques as in the previous subsections, we can analyze the 1-Ins channel for $m \triangleq n + 1$.

Theorem 1. If $R \in [m]$ and $r \equiv m \pmod R$, then

$$\max_{\mathbf{y} \in \Sigma_{q,R}^m} H_{1\text{-Ins}}^n(\mathbf{y}) = \log(m) - \frac{1}{m} \left(r \left\lfloor \frac{m}{R} \right\rfloor \log \left\lfloor \frac{m}{R} \right\rfloor + (R-r) \left\lfloor \frac{m}{R} \right\rfloor \log \left\lfloor \frac{m}{R} \right\rfloor \right),$$

and the maximum is obtained only by balanced channel outputs. Furthermore, $\max_{1\text{-Ins}}^n(n) = \log(m)$ and is attained only by channel outputs with m runs.

Theorem 2. If $R \in [m]$, then

$$\min_{\mathbf{y} \in \Sigma_{q,R}^m} H_{1\text{-Ins}}^n(\mathbf{y}) = \log(m) - \frac{(m-R+1) \log(m-R+1)}{m}$$

and the minimum is obtained only by skewed channel outputs. Furthermore, $\min_{1\text{-Ins}}^n(n) = 0$ and is attained only by channel outputs with a single run.

D. The Double-Deletion Channel: Minimum Input Entropy

This section studies the channel outputs $\mathbf{y} \in \Sigma_{2,R}^m$ that minimize the input entropy of the 2-Del channel. Let $\mathcal{S} \subseteq I_k(\mathbf{y})$, and define $W_{\mathcal{S}}(\mathbf{y}) \triangleq \sum_{\mathbf{x} \in \mathcal{S}} \omega_{\mathbf{y}}(\mathbf{x}) \cdot \log(\omega_{\mathbf{y}}(\mathbf{x}))$. From Lemma 2, for $n = k + m$ we know that

$$H_{k\text{-Del}}^n(\mathbf{y}) = \log\left(\binom{n}{k} 2^k\right) - \frac{1}{\binom{n}{k} 2^k} \sum_{\mathbf{x} \in I_k(\mathbf{y})} \omega_{\mathbf{y}}(\mathbf{x}) \cdot \log(\omega_{\mathbf{y}}(\mathbf{x})).$$

Therefore, $\arg \min_{\mathbf{y} \in \Sigma_{2,R}^m} H_{k\text{-Del}}^n(\mathbf{y}) = \arg \max_{\mathbf{y} \in \Sigma_{2,R}^m} W_{I_k(\mathbf{y})}(\mathbf{y})$. Let $\mathbf{y} \in \Sigma_{2,R}^m$ with $\mathcal{RL}(\mathbf{y}) = (r_1, r_2, \dots, r_R)$. For $i \in [0, R-1]$, let f_i denote the smallest index in $[i+1, R]$ such that $r_{f_i} > 1$, and let $f_R = R$. If such an index does not exist then let

$f_i = R$. Similarly, for $i \in [1, R]$, let b_i denote the largest index in $[1, i]$ such that $r_{b_i} > 1$, and let $b_0 = 1$. If such an index does not exist then let $b_i = 1$.

Lemma 6. Let $\mathbf{y} \in \Sigma_{2,R}^m$, such that $\mathcal{RL}(\mathbf{y}) = (r_1, \dots, r_R)$ and let $\mathbf{x} \in I_2(\mathbf{y})$. If $\mathcal{RL}(\mathbf{x}) = (r_1, \dots, r_i, 1, 1, r_{i+1}, \dots, r_R)$, then $\omega_{\mathbf{y}}(\mathbf{x}) = 1 + \sum_{j=b_i}^{f_i} r_j$.

Proof. In [4], $\omega_{\mathbf{y}}(\mathbf{x})$ was characterised as $r_i + r_{i+1} + 1$. In the case where $r_i = 1$ or $r_{i+1} = 1$ the analysis is a bit harder and the characterization is slightly different as follows. Let \mathbf{z} denotes the subsequence of \mathbf{y} from the b_i -th run to the f_i -th run. Note that \mathbf{z} is an alternating segment of length $\sum_{j=b_i}^{f_i} r_j$. Since \mathbf{x} is obtained from \mathbf{y} by adding two runs of length one after the i -th run, we can consider the subsequence \mathbf{z}' of \mathbf{x} which is the sequence \mathbf{z} with these additional two runs. It can be verified that deleting any two bits in \mathbf{x} that do not belong to \mathbf{z}' will result with a sequence different then \mathbf{y} . Therefore, $\omega_{\mathbf{y}}(\mathbf{x}) = \omega_{\mathbf{z}}(\mathbf{z}')$. Observed that $\omega_{\mathbf{z}}(\mathbf{z}') = |\mathbf{z}'| + 1$ and hence, $\omega_{\mathbf{y}}(\mathbf{x}) = 1 + \sum_{j=b}^f r_j$. \square

Let $\alpha \in \Sigma_2$, $\mathbf{x} \in \Sigma_2^n$, and denote by $\alpha \circ \mathbf{x}$ their concatenation. By abuse of notation, given a set of sequences \mathcal{S} , we let $\alpha \circ \mathcal{S} \triangleq \{\alpha \mathbf{x} : \mathbf{x} \in \mathcal{S}\}$. Next, we show how $\omega_{\mathbf{y}}(\mathbf{x})$ is effected when a bit α is appended to the beginning of both \mathbf{x} and \mathbf{y} .

Lemma 7. Let $\mathbf{y} \in \Sigma_2^m$ and $\mathbf{x} \in \Sigma_2^n$ with $m = n - k$. Let $\mathbf{y}' = \alpha \circ \mathbf{y}$ and $\mathbf{x}' = \alpha \circ \mathbf{x}$, then

$$\omega_{\mathbf{y}'}(\mathbf{x}') = \omega_{\mathbf{y}}(\mathbf{x}) + \sum_{i=1}^k \omega_{\mathbf{y}}(\mathbf{x}_{[i+1,n]}) \cdot \mathbb{I}_{\alpha=x_i},$$

where \mathbb{I} denotes an indicator function.

Proof. It can be verified that

$$\omega_{\mathbf{y}}(\mathbf{x}) = \begin{cases} \omega_{\mathbf{y}_{[2,m]}}(\mathbf{x}_{[2,n]}) + \omega_{\mathbf{y}}(\mathbf{x}_{[2,n]}) & y_1 = x_1 \\ \omega_{\mathbf{y}}(\mathbf{x}_{[2,n]}) & y_1 \neq x_1 \end{cases} \quad (3)$$

Therefore, $\omega_{\mathbf{y}'}(\mathbf{x}') = \omega_{\mathbf{y}}(\mathbf{x}) + \omega_{\mathbf{y}'}(\mathbf{x})$. Using (3) recursively, it follows that $\omega_{\mathbf{y}'}(\mathbf{x}') = \sum_{i=1}^k \omega_{\mathbf{y}}(\mathbf{x}_{[i+1,n]}) \cdot \mathbb{I}_{\alpha=x_i}$. \square

For $\mathbf{y} \in \Sigma_2^m$ with $\mathcal{RL}(\mathbf{y}) = (r_1, \dots, r_R)$, let $\mathbf{x}_{\beta}^{\mathbf{y}}, \mathbf{x}_{\gamma}^{\mathbf{y}}, \mathbf{x}_{\delta}^{\mathbf{y}} \in I_2(\mathbf{y})$, such that $\mathcal{RL}(\mathbf{x}_{\beta}^{\mathbf{y}}) \triangleq (1, 1, r_1, r_2, \dots, r_R)$, $\mathcal{RL}(\mathbf{x}_{\gamma}^{\mathbf{y}}) \triangleq (2, r_1, r_2, \dots, r_R)$ and $\mathcal{RL}(\mathbf{x}_{\delta}^{\mathbf{y}}) \triangleq (1, r_1 + 1, r_2, \dots, r_R)$. The next two corollaries follows from Lemma 7.

Corollary 4. Let $\mathbf{y} \in \Sigma_2^{n_1}$ it holds that:

- 1) $\omega_{\mathbf{y}}(\mathbf{x}_{\beta}^{\mathbf{y}}) = 1 + \sum_{j=1}^{f_0} r_j$, which is maximal for $f_0 = \rho(\mathbf{y})$.
- 2) $\omega_{\mathbf{y}}(\mathbf{x}_{\gamma}^{\mathbf{y}}) = 1$.
- 3) $\omega_{\mathbf{y}}(\mathbf{x}_{\delta}^{\mathbf{y}}) = r_1 + 1$, which is maximal for $\rho(\mathbf{y}) = 1$ and minimal for $\rho(\mathbf{y}) = m$.

Corollary 5. Let $\mathbf{y} \in \Sigma_2^m$, $\mathbf{x} \in \Sigma_2^{m+2}$ and let $\mathbf{y}' = \alpha \circ \mathbf{y}$, $\mathbf{x}' = \alpha \circ \mathbf{x}$. If $\alpha \neq y_1$ or $y_1 \neq x_1$, then,

$$\omega_{\mathbf{y}'}(\mathbf{x}') = \begin{cases} \omega_{\mathbf{y}}(\mathbf{x}) + \mathbb{I}_{\mathbf{x}=\mathbf{x}_{\beta}^{\mathbf{y}}} & \alpha \neq y_1, y_1 = x_1 \\ 2 \cdot \omega_{\mathbf{y}}(\mathbf{x}) + \mathbb{I}_{\mathbf{x}=\mathbf{x}_{\gamma}^{\mathbf{y}}} & \alpha \neq y_1, y_1 \neq x_1 \\ \omega_{\mathbf{y}}(\mathbf{x}) + \mathbb{I}_{\mathbf{x}=\mathbf{x}_{\delta}^{\mathbf{y}}} & \alpha = y_1, y_1 \neq x_1, \end{cases}$$

Let $\sigma^{\ell} \in \Sigma_2^{\ell}$ be a constant word of length ℓ , i.e., $\rho(\sigma^{\ell}) = 1$. The next lemma shows that $\mathbf{y} = \sigma^{\ell}$ minimizes $H_{2\text{-Del}}^{\text{In}}(\mathbf{y})$.

Lemma 8. $H_{2\text{-Del}}^{\text{In}}(\mathbf{y})$ is minimized only by σ^m and

$$\min_{2\text{-Del}}^{\text{In}}(n) = H_{2\text{-Del}}^{\text{In}}(\sigma^m) = 2 + \frac{3}{4} \log \binom{m}{2} - \frac{1}{2} \log(m+1).$$

Proof. We prove the lemma using induction on m . Let

$$\mathbf{g}_m \triangleq \arg \max_{\mathbf{y} \in \Sigma_2^m} W_{y_1 \circ I_2(\mathbf{y}_{[2,m]})}(\mathbf{y}). \quad (4)$$

For the base case it can be verified that $\mathbf{g}_2 = \{00, 11\}$. For $\ell < m$, assume that $\mathbf{g}_{\ell-1} = \sigma^{\ell-1}$. From (3), $\omega_{\mathbf{y}}(\overline{y_1} \circ \mathbf{x}) = \omega_{\mathbf{y}}(\mathbf{x})$, and by Corollary 3 it can be deduced that,

$$\arg \max_{\mathbf{y} \in \Sigma_2^{\ell-1}} W_{\overline{y_1} \circ I_1(\mathbf{y})}(\mathbf{y}) = \sigma^{\ell-1}. \quad (5)$$

We will now show that $\mathbf{g}_{\ell} = \sigma^{\ell}$. Let $\mathbf{y} \in \Sigma_2^{\ell}$, $\mathcal{RL}(\mathbf{y}_{[2,\ell]}) \triangleq (r_1, r_2, \dots, r_R)$, $\mathbf{x} \in I_2(\mathbf{y}_{[2,\ell]})$. Consider the next two cases,

Case I - $y_1 \neq y_2$: From Corollary 5, we know that

$$\omega_{\mathbf{y}}(y_1 \circ \mathbf{x}) = \begin{cases} \omega_{\mathbf{y}_{[2,\ell]}}(\mathbf{x}) + \mathbb{I}_{\mathbf{x}=\mathbf{x}_{\beta}^{\mathbf{y}}} & y_2 = x_1 \\ 2 \cdot \omega_{\mathbf{y}_{[2,\ell]}}(\mathbf{x}) + \mathbb{I}_{\mathbf{x}=\mathbf{x}_{\gamma}^{\mathbf{y}}} & y_2 \neq x_1. \end{cases}$$

Next using Corollary 4, (4) and (5), it can be verified that

$$\arg \max_{y_1 \neq y_2, \mathbf{y}_{[2,\ell]} \in \Sigma_2^{\ell-1}} W_{y_1 \circ I_2(\mathbf{y}_{[2,\ell]})}(\mathbf{y}) = \sigma_1^{\ell} \circ \sigma^{\ell-1}.$$

Case II - $y_1 = y_2$: By Corollary 5, if $y_2 \neq x_1$, then $\omega_{\mathbf{y}}(y_1 \circ \mathbf{x}) = \omega_{\mathbf{y}}(\mathbf{x}) + \mathbb{I}_{\mathbf{x}=\mathbf{x}_{\beta}^{\mathbf{y}}}$. Otherwise, if $y_2 = x_1$, we define $\omega_{\mathbf{y}'}(\mathbf{x}) \triangleq f(r_1, r_2, \dots, r_R, \mathbf{x})$, then $\omega_{\mathbf{y}}(y_1 \circ \mathbf{x}) = f(r_1 + 1, r_2, \dots, r_R, \mathbf{x})$, see [43] for more details. Using Corollary 4, (4) and (5), it can be verified that

$$\arg \max_{y_1=y_2, \mathbf{y}_{[2,\ell]} \in \Sigma_2^{\ell-1}} W_{y_1 \circ I_2(\mathbf{y}_{[2,\ell]})}(\mathbf{y}) = \sigma^{\ell}.$$

The following results can be obtained by a manual comparison of these two cases.

$$\arg \max_{\mathbf{y} \in \Sigma_2^{\ell}} W_{I_2(\mathbf{y})}(\mathbf{y}) = \arg \max_{\mathbf{y} \in \Sigma_2^{\ell}} (W_{y_1 \circ I_2(\mathbf{y}_{[2,\ell]})}(\mathbf{y}) + W_{\overline{y_1} \circ I_1(\mathbf{y})}(\mathbf{y})) = \sigma_1^{\ell}.$$

\square

V. THE AVERAGE INPUT ENTROPY

Let N_r^m be the number of runs with length r over all the sequences in Σ_q^m . It can be verified that

$$N_r^m = (q-1)q^{m-r-1} ((q-1)(m-r+1) + 2)$$

We now derive the expressions for the average input entropies for the 1-deletion channel and the 1-insertion channel.

Lemma 9. For any n, q we have that,

$$\text{avg}_{1\text{-Del}}^{\text{In}}(n) = \log(nq) - \frac{q-1}{n} \sum_{r=1}^{n-1} \frac{((q-1)(n-r)+2)(r+1) \log(r+1)}{q^{r+2}}$$

$$\text{avg}_{1\text{-Ins}}^{\text{In}}(n) = \log(n+1) - \frac{q-1}{n+1} \sum_{r=1}^{n+1} \frac{((q-1)(n-r+2)+2) r \log(r)}{q^{r+1}}$$

Clearly, $\text{avg}_{1\text{-Del}}^{\text{In}}(n) \leq \max_{1\text{-Del}}^{\text{In}}(n)$ and $\text{avg}_{1\text{-Ins}}^{\text{In}}(1) \leq \max_{1\text{-Ins}}^{\text{In}}(n)$ for any integers n, q . To improve the lower bounds on the average entropies which are better than the minimum values, note that if $r \geq 1$, then $r \geq \log(r+1)$. Hence by Lemma 9, it can be shown that,

$$\text{avg}_{1\text{-Del}}^{\text{In}}(n) \geq \log(nq) - \frac{1}{n} \left(\frac{2n}{q-1} - \frac{n^2-n}{q^{n+1}} + \frac{2q^2-2q^{n+2}}{(q-1)^2 q^{n+1}} \right)$$

$$\text{and } \text{avg}_{1\text{-Ins}}^{\text{In}}(n) \geq \log(nq) + \frac{1}{n+1} \left(\frac{n^2}{q^{n+2}} - \frac{n(2q^{n+2}-q+1)}{(q-1)q^{n+2}} + \frac{2(q^n-1)}{(q-1)^2 q^n} \right).$$

The bounds for $\text{avg}_{1\text{-Del}}^{\text{In}}(n)$ are presented in Figure 1.

REFERENCES

- [1] M. Abroshan, R. Venkataramanan, L. Dolecek, and A. G. i Fàbregas, "Coding for deletion channels with multiple traces," *Proc. Int. Symp. Inf. Theory*, pp. 1372–1376, 2019.
- [2] M. Abu-Sini, and E. Yaakobi, "On list decoding of insertions and deletions under the reconstruction model," *Proc. Int. Symp. Inf. Theory*, pp. 1706–1711, 2021.
- [3] L. Anavy, I. Vaknin, O. Atar, R. Amit, and Z. Yakhini, "Data storage in DNA with fewer synthesis cycles using composite DNA letters," *Nature biotechnology*, vol. 37, no. 10, pp. 1229–1236, 2019.
- [4] A. Atashpendar, M. Beunardeau, A. Connolly, R. Géraud, D. Mestel, A. W. Roscoe, and P. Y. A. Ryan, "From clustering supersequences to entropy minimizing subsequences for single and double deletions," *arXiv:1802.00703*, 2018.
- [5] A. Atashpendar, A. Roscoe, and P.Y. Ryan, "Information leakage due to revealing randomly selected bits," *Security Protocols XXIII. Springer*, pp. 325–341, 2015.
- [6] D. Bar-Lev, I. Orr, O. Sabary, T. Etzion, and E. Yaakobi, "Deep DNA storage: Scalable and robust DNA storage via coding theory and deep learning," *arXiv preprint arXiv:2109.00031*, 2021.
- [7] D. Bar-Lev, Y. Gershon, O. Sabary and E. Yaakobi, "Decoding for optimal expected normalized distance over the t -deletion channel," *Proc. Int. Symp. Inf. Theory*, pp. 1847–1852, 2021.
- [8] T. Batu, S. Kannan, S. Khanna, and A. McGregor, "Reconstructing strings from random traces," *ACM-SIAM symposium on Discrete algorithms*, pp. 910–918. Society for Industrial and Applied Mathematics, 2004.
- [9] J. Brakensiek, V. Guruswami, and S. Zbarsky, "Efficient low-redundancy codes for correcting multiple deletions," *Annual ACM-SIAM Symp. on Discrete Algorithms*, pp. 1884–1892, Philadelphia, PA, USA, 2016.
- [10] J. Sima and J. Bruck, "Trace reconstruction with bounded edit distance," *Int. Symp. Inf. Theory*, pp. 2519–2524, 2021.
- [11] J. Castiglione and A. Kavcic, "Trellis based lower bounds on capacities of channels with synchronization errors," *Proc. IEEE Inf. Theory Workshop*, pp. 24–28, Jeju, South Korea, 2015.
- [12] M. Cheraghchi, J. Downs, J. Ribeiro and A. Veliiche, "Mean-Based Trace Reconstruction over Practically any Replication-Insertion Channel," *Proc. Int. Symp. Inf. Theory*, pp. 2459–2464, 2021
- [13] S. Davies, M. Z. Rácz, B. G. Schiffer and C. Rashtchian, "Approximate Trace Reconstruction: Algorithms," *Proc. Int. Symp. Inf. Theory*, pp. 2525–2530, 2021.
- [14] L. Dolecek and V. Anantharam, "Using Reed Muller $RM(1, m)$ codes over channels with synchronization and substitution errors," *IEEE Trans. on Inform. Theory*, vol. 53, no. 4, pp. 1430–1443, 2007.
- [15] R. L. Dobrushin, "Shannon's theorems for channels with synchronization errors," *Problemy Peredachi Informatsii*, vol. 3, no. 4, pp. 18–36, 1967.
- [16] R. Gabrys and F. Sala, "Codes correcting two deletions," *IEEE Trans. on Inform. Theory*, vol. 65, no. 2, pp. 965–974, 2018.
- [17] R. Gabrys and E. Yaakobi, "Sequence reconstruction over the deletion channel," *IEEE Trans. on Inform. Theory*, vol. 64, no. 4, pp.2924–2931, 2018.
- [18] B. Graham, "A Binary Deletion Channel With a Fixed Number of Deletions," *Combinatorics, Probability and Computing*, vol. 24, no. 3, pp. 486–489, 2018.
- [19] R. N. Grass, R. Heckel, M. Puddu, D. Paunescu, and W. J. Stark, "Robust chemical preservation of digital information on DNA in silica with error-correcting codes," *Angewandte Chemie International Edition*, vol. 54, no. 8, pp. 2552–2555, 2015.
- [20] V. Guruswami, B. Haeupler, and A. Shahrashi, "Optimally resilient codes for list-decoding from insertions and deletions," *Proc. of the 52nd Annual ACM SIGACT Sym. on Theory of Computing*, pp. 524–537, 2020.
- [21] V. Guruswami and C. Wang, "Deletion codes in the high-noise and high-rate regimes," *IEEE Trans. Inf. Theory*, vol. 63, no. 4, pp. 1961–1970, 2017.
- [22] T. Hayashi and K. Yasunaga, "On the list decodability of insertions and deletions," *Proc. Int. Symp. Inf. Theory*, pp. 86–90, 2018.
- [23] R. Heckel, G. Mikutis, and R.N. Grass, "A characterization of the DNA data storage channel," *Scientific Reports*, vol. 9, pp. 9663, 2019.
- [24] N. Holden, R. Pemantle, and Y. Peres, "Subpolynomial trace reconstruction for random strings and arbitrary deletion probability," *arXiv preprint arXiv:1801.04783*, 2018.
- [25] S. Kas Hanna and S. El Rouayheb, "List decoding of deletions using guess & check codes," *Proc. Int. Symp. Inf. Theory*, pp. 2374–2378, 2019.
- [26] K. Cai, H. M. Kiah, T. T. Nguyen, and E. Yaakobi, "Coding for sequence reconstruction for single edits," *IEEE Trans. Inf. Theory*, 2021.
- [27] A. Kirsch and E. Drinea, "Directly lower bounding the information capacity for channels with i.i.d. deletions and duplications," *IEEE Trans. on Info. Theory*, vol. 56, no. 1, pp. 86–102, 2010.
- [28] V. I. Levenshtein, "Binary codes capable of correcting deletions, insertions, and reversals," *Soviet Physics Doklady*, vol. 10, no. 8, pp. 707–710, 1966.
- [29] V. I. Levenshtein, "Efficient reconstruction of sequences," *IEEE Tans. on Info. Theory*, vol. 47, no. 1, pp. 2–22, 2001.
- [30] V. I. Levenshtein, "Efficient reconstruction of sequences from their subsequences or supersequences," *Journal of Combinatorial Theory, Series A*, vol. 93, no. 2, pp.310–332, 2001.
- [31] S. Liu, I. Tjuawinata, and C. Xing, "On list decoding of insertion and deletion errors," <https://arxiv.org/abs/1906.09705>, 2019.
- [32] M. Mitzenmacher and E. Drinea, "A simple lower bound for the capacity of the deletion channel," *IEEE Trans. on Info. Theory*, vol. 52, no. 10, pp. 4657–4660, 2006.
- [33] L. Organick et al., "Random access in large-scale DNA data storage," *Nature Biotechnology*, vol. 36, no. 3, pp. 242–248, 2018.
- [34] C. Pan et al., "Rewritable Two-Dimensional DNA-Based Data Storage with Machine Learning Reconstruction," *bioRxiv 2021.02.22.432304*, 2021.
- [35] M. Rahmati and T. M. Duman, "Upper bounds on the capacity of deletion channels using channel fragmentation," *IEEE Trans. of Info. Theory*, vol. 61, no. 1, pp. 146–156, 2015.
- [36] O. Sabary, D. Bar-Lev, Y. Gershon, A. Yucovich, and E. Yaakobi, "On the decoding error weight of one or two deletion channels," *arXiv:2201.02466*, 2022.
- [37] O. Sabary, A. Yucovich, G. Shapira, and E. Yaakobi, "Reconstruction algorithms for DNA-storage systems," *bioRxiv 2020.09.16.300186*.
- [38] O. Sabary, Y. Orlev, R. Shafir, L. Anavy, E. Yaakobi, and Z.Yakhini, "SOLQC: Synthetic oligo library quality control Tool," *Bioinformatics*, vol. 37, no. 5, pp. 720–722, 2021.
- [39] F. Sala, C. Schoeny, N. Bitouzé, and L. Dolecek, "Synchronizing files from a large number of insertions and deletions," *IEEE Trans. on Comm.*, vol. 64, no. 6, pp. 2258–2273, 2016.
- [40] C. Schoeny, A. Wachter-Zeh, R. Gabrys, and E. Yaakobi, "Codes correcting a burst of deletions or insertions," *IEEE Trans. on Info. Theory*, vol. 63, no. 4, pp. 1971–1985, 2017.
- [41] J. Sima and J. Bruck, "Optimal k -deletion correcting codes," *Proc. Int. Symp. Inf. Theory*, pp. 847–851, 2019.
- [42] J. Sima, N. Raviv, and J. Bruck, "On coding over sliced information," *IEEE Trans. on Info. Theory*, vol. 67, no. 5, pp. 2793–2807, 2021.
- [43] S. Singhvi, O. Sabary, D. Bar-Lev, and E. Yaakobi, "The Input and Output Entropies of the k -Deletion/Insertion Channel," *arXiv preprint arXiv:2202.03024*, 2022.
- [44] S. R. Srinivasavaradhan, S. Gopi, H. Pfister, S. Yekhanin, "Trellis BMA: Coded Trace Reconstruction on IDS Channels for DNA Storage," *Proc. Int. Symp. Inf. Theory*, pp. 2453–2458, 2021.
- [45] S. Tabatabaei et al., "DNA punch cards for storing data on native DNA sequences via enzymatic nicking," *Nature communications*, vol. 11, no. 1, pp. 1–10, 2020.
- [46] R. R. Varshamov and G. M. Tenenholtz, A code for correcting a single asymmetric error, *Automatica i Telemekhanika*, vol. 26, no. 2, pp. 288–292, 1965.
- [47] A. Wachter-Zeh, "List decoding of insertions and deletions," *IEEE Trans. of Info. Theory*, vol. 64, no. 9, pp. 6297–6304, 2017.
- [48] E. Yaakobi, M. Schwartz, M. Langberg, and J. Bruck, "Sequence reconstruction for grassmann graphs and permutations," *Proc. Int. Symp. Inf. Theory*, pp. 874–878, 2013.
- [49] S. H. T. Yazdi, R. Gabrys, and O. Milenkovic, "Portable and error-free DNA-based data storage," *Scientific Reports*, vol. 7, no. 1, pp. 1–6, 2017.
- [50] D. Bar-Lev, T. Etzion and E. Yaakobi, "On Levenshtein Balls with Radius One," 2021 IEEE International Symposium on Information Theory (ISIT), Melbourne, Australia, 2021, pp. 1979–1984, doi: 10.1109/ISIT45174.2021.9517922.
- [51] D. Bar-Lev, T. Etzion and E. Yaakobi, "On the Size of Balls and Anticodes of Small Diameter under the Fixed-Length Levenshtein Metric," in *IEEE Transactions on Information Theory*, doi: 10.1109/TIT.2022.3227128.

[52] N. J. Sloane (2002), On Single-Deletion-Correcting Codes.