



PIR Codes with Short Block Length

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Abstract

In this work *private information retrieval (PIR) codes* are studied. In a k -PIR code, s information bits are encoded in such a way that every information bit has k mutually disjoint recovery sets. The main problem under this paradigm is to minimize the number of encoded bits given the values of s and k , where this value is denoted by $P(s, k)$. The main focus of this work is to analyze $P(s, k)$ for a large range of parameters of s and k . In particular, we improve upon several of the existing results on this value.

Keywords Private information retrieval · PIR codes · Finite projective geometry · Linear codes

Mathematics Subject Classification Primary 68P30 · Secondary 05D99

1 Introduction

A k -private information retrieval (k -PIR) code is a coding scheme which encodes by linear operations some s information bits to n encoded bits such that each information bit has k mutually disjoint *recovery sets*. The main figure of merit when studying PIR codes is the length n of the code, given the values of s and k . Thus, the value $P(s, k)$ denotes the minimum value of n for which a length- n k -PIR code exists.

PIR codes were studied recently in [10] due to their applicability for private information retrieval in a coded storage system [7]. They were beneficial in reducing the storage overhead by partitioning the database into some s parts and distribute these parts among several non-communicating servers. Given a partition of the database into s parts, the construction from [10] consisted of two main ingredients: 1) an existing k -server PIR protocol in which the servers' responses are a linear function of the database bits, and 2) a k -PIR code. The first

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requirement was very easy to satisfy as almost all known existing PIR protocols are linear in this fashion. Thus the primary focus has been given to the construction of k -PIR codes. Note that PIR codes can only be used in the PIR setup together with existing PIR protocols, and as such, they do not provide independently a PIR protocol. However, we use the term PIR codes as this is the common one in the literature for this family of codes [1, 10, 18, 20, 25, 27, 28, 30]. In fact, PIR codes are similar in their definition to *locally repairable codes (LRCs) with availability* [22, 26, 29], however PIR codes do not impose any constraint on the size of the recovery sets as done for LRCs. Furthermore, these codes have more in common with *one-step majority-logic decodable codes* that were studied a while ago by Massey [21] and later by Lin and others [17] for applications of fast decoding. The main difference is that one-step majority-logic decodable codes require that each symbol (both information and redundancy) will have multiple recovery sets.

The rest of this paper is organized as follows. In Sect. 2, we formally define the codes studied in the paper, list some of the known previous results which are relevant to our work, and discuss several preliminary results. In Sect. 3, it is shown how to construct PIR codes by lengthening existing codes. Additionally we give a geometric construction using s -dimensional simplex codes as a starting point. In Sect. 4, we present a general linear programming formulation for PIR codes which provides lower bounds on the parameters of these codes and in many cases completely determine the value of $P(s, k)$. Coding theoretic methods are applied in Sect. 5 in order to obtain a few more lower bounds and exact values. We fully solve the cases where $s = 4$ or $s = 5$ and present further lower and upper bounds in Sect. 6. There we also summarize the best known lower and upper bounds for small parameters in Table 1.

2 Definitions, Previous Work, and Preliminaries

2.1 Definitions

In this section we formally define the codes studied in this paper. A binary linear code of length n and dimension s will be denoted by $[n, s]$ or $[n, s, d]$, where d denotes its minimum Hamming distance. The set $[n]$ denotes the set of integers $\{1, 2, \dots, n\}$. The binary field is denoted by \mathbb{F}_2 .

In this work we focus on PIR codes that were defined recently in [11]. This family of codes requires to encode some s information bits into n encoded bits such that every information bit has k mutually disjoint recovery sets. Formally, these codes are defined as follows.

Definition 1 An $[n, s]$ binary linear code \mathcal{C} will be called a k -PIR code, and will be denoted by $[n, s, k]_P$, if for every unit vector e_i (corresponding to an information bit), $i \in [s]$, there exist k mutually disjoint sets $R_{i,1}, \dots, R_{i,k} \subseteq [n]$ such that e_i is the sum of the columns of a generator matrix of \mathcal{C} whose indices belong to $R_{i,j}$ ¹.

The main problem in studying PIR codes is to minimize the length n given the values of s and k . We denote by $P(s, k)$ the smallest n such that there exists an $[n, s, k]_P$ code and the optimal redundancy of k -PIR codes is defined by $r_P(s, k) \triangleq P(s, k) - s$. In case $k = 1$, the code $[s, s]$ which simply stores all the information symbols is an $[s, s, 1]_P$ PIR code, so that $P(s, 1) = s$. Similarly, the simple parity check code $[s + 1, s]$ is an $[s + 1, s, 2]_P$ PIR code which implies that $P(s, 2) = s + 1$.

¹ Note that the definition of a k -PIR code actually depends on the representing generator matrix and not the code itself.

2.2 Previous Work

In [11], it was shown that for any fixed $k \geq 3$ it is possible to construct $[n, s, k]_P$ PIR codes where $n = s + \mathcal{O}(\sqrt{s})$, so $r_P(s, k) = \mathcal{O}(\sqrt{s})$ for any fixed $k \geq 3$, and in [25,30] it was proved that $r_P(s, 3) = \Omega(\sqrt{s})$. Since $r_P(s, k) \geq r_P(s, 3)$ for any fixed $k \geq 3$, these results assure also that for any fixed $k \geq 3$, $r_P(s, k) = \Theta(\sqrt{s})$.

There are several known results and constructions of PIR codes; see e.g. [12,18,28]. We summarize here the most relevant known results for our problem:

- Theorem 2**
1. $P(s_1 + s_2, k) \leq P(s_1, k) + P(s_2, k)$, [11].
 2. $P(s, k_1 + k_2) \leq P(s, k_1) + P(s, k_2)$, [11].
 3. $P(s, 2k) = P(s, 2k - 1) + 1$, [11].
 4. $r_P(s, k) = \Theta(\sqrt{s})$ for fixed k , [11,25,30].
 5. $P(s, 2^{s-1}) = 2^s - 1$, [11].
 6. For every integer $k \geq 1$ we have $P(2, k) = \lceil 3k/2 \rceil$, [11].
 7. For every even integer $k \geq 2$ we have $P(3, k) = \lceil 7k/4 \rceil$, [11].
 8. For every positive s and k , $P(s, k) \geq \frac{2^s - 1}{2^{s-1}} k$ and equality if and only if k is a multiple of 2^{s-1} , [11, Theorem 18]

Additional constructions of PIR codes from bipartite graphs and constant weight codes were stated in [11, Section IV.D]. One-step majority logic codes were used to obtain the following result.

Theorem 3 ([10, Theorem 9],[11, Theorem 8]) For any θ, ℓ , and λ we have

$$P\left(2^{2\theta\ell} - (2^{\theta+1} - 1)^\ell - 1, 2^\ell + 2\right) \leq 2^{2\theta\ell} - 1.$$

$$P\left((2^\lambda - 1)^\ell - 2, 2^\ell\right) \leq 2^{\lambda\ell} - 1.$$

We remark that the formulations of [10, Theorem 9] and [11, Theorem 8] have been slightly adjusted by reducing the value of s by one. As already observed in [18], applying $\theta = 1, \ell = 2$ in the first formula of [10, Theorem 9] or [11, Theorem 8] gives $P(7, 6) \leq 15$. However, $P(7, 6) \geq N(7, 6) = 16$, see (1), which is a contradiction. In our version we obtain $P(6, 6) \leq 15$. In [18, Section V, Remark 3], the authors traced the problem back to a misprint in [8, p. 289]; see [19, Theorem 6] for the correct version.

An analytic solution of the general lengthening problem, see the discussion in Sect. 2.3, resulted with the following.

Theorem 4 ([18, Theorem 1]) $P(s + 1, k) \leq P(s, k) + \lceil \frac{k}{2} \rceil$.

Since the minimum Hamming distance of every k -PIR code is at least k it holds that

$$P(s, k) \geq N(s, k), \tag{1}$$

see e.g. [18, Corollary 1] or [20, Theorem 8], where $N(s, k)$ denotes the smallest integer n such that an $[n, s, k]$ code exists. In [15], the so-called Griesmer bound

$$N(s, k) \geq \sum_{i=0}^{s-1} \left\lceil \frac{k}{2^i} \right\rceil =: G(s, k) \geq \frac{2^s - 1}{2^{s-1}} \cdot k \tag{2}$$

was proven. Interestingly enough, for every fixed integer s we have $N(s, k) = G(s, k)$ if k is sufficiently large [2], i.e., for every fixed integer s the determination of the function $N(s, k)$ is a finite problem. Those functions are explicitly known for all $s \leq 8$ [4]. Several works

have been devoted to the determination of $N(s, k)$ for specific parameters s and k . For the currently best known lower and upper bounds on $N(s, k)$ we refer to the online tables [14].² The values of $N(s, k)$ are a good benchmark for constructions of s -dimensional binary k -PIR codes, i.e., constructive upper bounds for $P(s, k)$. If $N(s, k)$ is met, then the corresponding PIR code is obviously optimal.

It is quite hard to get better lower bounds than $P(s, k) \geq N(s, k)$. One parametric improvement was stated in the literature so far: $P(s, 3) \geq s + \left\lceil \sqrt{2s + \frac{1}{4} + \frac{1}{2}} \right\rceil$, see [25, Theorem 3, Equation 10]. If we combine it with the puncturing constraint $P(s, k) \geq P(s, k - 1) + 1$, then we obtain

$$P(s, k) \geq \left\lceil \sqrt{2s + \frac{1}{4} + \frac{1}{2}} \right\rceil + k - 3 \tag{3}$$

for $k \geq 3$. We remark that for $k = 3$ or $k = 4$ this inequality is always at least as good as the coding theoretic lower bound $P(s, k) \geq N(s, k)$ and it is indeed a strict improvement for larger values of s .³ For systematic PIR codes, see the explanation below, the same bound was also proved in [28] and [30].

Despite significant progress on determining the exact value of $P(s, k)$, this problem is far from being solved. The goal of this paper is to build upon previous work and develop new tools which are specifically targeted towards deriving upper and lower bound on $P(s, k)$ in order to establish many cases which still remained open. For example, if we apply Theorem 4 to a 9-dimensional 10-PIR code of length 28, we can conclude $P(10, 10) \leq 33$, which improves the best known construction. By solving an instance of the general lengthening problem this can even be improved to $P(10, 10) \leq 31$, see the discussion in Sect. 2.3.

2.3 Preliminaries

Note that an $[n, s]$ code is a k -PIR code if it admits a generator matrix $G \in \mathbb{F}_2^{s \times n}$ such that for each $1 \leq i \leq s$ there exist disjoint sets $R_1^i, \dots, R_k^i \subseteq [n]$ such that $\sum_{h \in R_j^i} G^h = e_i$ for all $1 \leq j \leq k$, where G^h denotes the h th column of G and e_i denotes the i th unit vector. The interpretation is that e_i can be recovered by the k disjoint sets R_j^i , which are therefore also called *recovery sets*. The set of all recovery sets for e_i is denoted by \mathcal{R}_i , i.e., $\mathcal{R}_i = \{R_j^i \mid 1 \leq j \leq k\}$. We call a recovery set R for e_i *minimal* if no proper subset $R' \subsetneq R$ is a recovery set for e_i .

Example 5 An example for a generator matrix attaining $P(4, 4) = 9$ is given by:

$$G = \begin{pmatrix} 100011111 \\ 010001011 \\ 001011001 \\ 000100111 \end{pmatrix}.$$

For e_4 we can use the recovery sets

$$\{4\}, \{1, 7\}, \{6, 9\}, \{2, 3, 5, 8\}.$$

² We remark that with respect to lower bounds on $N(s, k)$ it makes also sense to check the entries at <http://mint.sbg.ac.at> which sometimes contain improvements.

³ More precisely, it is a strict improvement for $s = 4$ and all $s \geq 7$. An exact formula for $N(s, 3)$, and so also for $N(s, 4)$, exists. It is attained by the Hamming codes and puncturings thereof. The lower bound follows from the Hamming or sphere packing bound.

Note that there is also a different list of recovery sets:

$$\{4\}, \{6, 9\}, \{3, 5, 7\}, \{1, 2, 8\}.$$

The later might have the advantage that it only uses recovery sets of cardinality at most 3. As a different notation for recovery sets we also use the columns directly (instead of their labels). In our last example \mathcal{R}_4 then reads

$$\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

For minimal recovery sets these are indeed sets of non-zero vectors in \mathbb{F}_2^s and multisets in general. In the latter case points can have a multiplicity larger than 1.

Each binary linear $[n, s]$ code C of effective length n , i.e., the number of non-zero columns in a generator matrix, is in bijection to a multiset \mathcal{P} of points, i.e., 1-dimensional subspaces of \mathbb{F}_2^s , of cardinality n . Starting from a generator matrix G we can obtain a multiset of points by choosing the points $\langle G^k \rangle$ for every column G . In the other direction we can choose an arbitrary generator for each point, i.e., a column vector, and build up a generator matrix with those column vectors. This geometrical point of view can give an easy way to form constructions of PIR codes. For example, by taking the set of all $2^s - 1$ points in \mathbb{F}_2^s , we get the so-called *s-dimensional simplex code* in order to get the known result of $P(s, 2^{s-1}) \leq 2^s - 1$ for all $s \geq 1$.⁴ We will use this geometric formulation when deriving several of our results, see especially Proposition 9. Note that the s unit vectors e_1, e_2, \dots, e_s form a basis that spans the s -dimensional space \mathbb{F}_2^s and in the sequel, we refer to an $(s - 1)$ -dimensional subspace as a *hyperplane*.

For a lower bound, let H be an arbitrary hyperplane of \mathbb{F}_2^s . Since $\dim(H) = s - 1$, it holds that $H \neq \mathbb{F}_2^s$ and thus there exists at least one index $1 \leq i \leq s$ such that $\langle e_i \rangle$ is not contained in H . Thus, there have to be at least k points outside of H to form the recovery sets:

Lemma 6 (C.f. [27, Theorem 2], [20, Lemma 2])

Let \mathcal{P} be the multiset of points corresponding to an s -dimensional k -PIR code. For every hyperplane H of \mathbb{F}_2^s we have

$$|\{P \in \mathcal{P} \mid P \notin H\}| \geq k, \tag{4}$$

while counting points with their respective multiplicities.

Summing up Inequality (4) for the $2^s - 1$ hyperplanes gives $P(s, k) \geq \frac{2^s - 1}{2^s - 1} \cdot k$, see [11, Theorem 16], since each point of \mathbb{F}_2^s is contained in exactly $2^{s-1} - 1$ hyperplanes. While this gives $P(s, 2^{s-1}) = 2^s - 1$, it can be improved easily. Since each hyperplane of \mathbb{F}_2^s corresponds to a codeword c_H of the code whose weight equals the number of points outside of H , we have the well known fact that the minimum Hamming distance of any k -PIR code is at least k . Thus, we have $P(s, k) \geq N(s, k)$.

We call a generator matrix G of a linear code systematic if it contains a unit matrix. While every linear code admits a systematic generator matrix it is not clear whether there always exists a systematic PIR code matching $P(s, k)$, see Question 4 in [27, Sec. 10]. Here we give an example that this is not the case. More specifically, we will show $P(6, 8) = 19$ in Sect.

⁴ As an abbreviation set $k = 2^{s-1}$ and number the 2^{s-1} vectors of \mathbb{F}_2^s with i th component equal to zero by $x^{1,i}, \dots, x^{k,i}$, where we assume that $x^{1,i}$ is the zero vector. For each $1 \leq i \leq s$ we can take the recovery sets $R_1^i = \{x^{1,i} + e_i\}$ and $R_j^i = \{x^{j,i}, x^{j,i} + e_i\}$ for $2 \leq j \leq k$.

4 while every systematic PIR code has length at least 20, see Proposition 11. An optimal non-systematic generator matrix (of length 19) is given by

$$\begin{pmatrix} 0101111011110111011 \\ 1111010111101101110 \\ 1111001111011111001 \\ 1100111110111100111 \\ 1100000011111111111 \\ 0011111111111100000 \end{pmatrix}$$

and an optimal systematic generator matrix (of length 20) is given by

$$\begin{pmatrix} 100000111111110011 \\ 0100001011011001111 \\ 0010011001111100000 \\ 0001001111001010000 \\ 0000100111000111111 \\ 0000100001111111111 \end{pmatrix}.$$

We call a code projective if all columns of an arbitrary generator matrix are pairwise linear independent, i.e., no column is a multiple of another one. Note that the last-but-one generator matrix (and so the code) is projective, while the last generator matrix is not, since the last four columns correspond to two points with multiplicity 2 each.

According to this observation we notice that while the objects under consideration are called PIR codes, in fact their properties actually can depend on their generator matrix. This is different for codes with disjoint repair groups as e.g. studied in [30]. The code property of disjoint repair groups is more demanding than that of PIR codes (depending on the generator matrix). Namely, while the property of PIR codes heavily depends on the choice of the generator matrix of the code (i.e., the same code can be a PIR code with one choice of a parity check matrix and not a PIR code with a different generator matrix), codes with disjoint repair groups [30] are properties of the code itself independently of the specific choice of the generator matrix. However, note that for systematic generator matrices both notions are the same, so that the lower bound from [30] only works (directly) for systematic generator matrices.

Let \mathcal{P} be a multiset of points in \mathbb{F}_2^s and n denote its cardinality $|\mathcal{P}|$. By h_i we denote the number of hyperplanes of \mathbb{F}_2^s that contain exactly i points. Counting incidences (H) , (H, P) , and (H, P, P') , where the H are hyperplanes and P, P' are different points in \mathcal{P} , gives the so-called standard equations

$$\sum_{i \geq 0} h_i = 2^s - 1 \tag{5}$$

$$\sum_{i \geq 0} i h_i = n (2^{s-1} - 1) \tag{6}$$

$$\sum_{i \geq 0} i(i - 1) h_i = n(n - 1) (2^{s-2} - 1) + 2^{s-2} y_2, \tag{7}$$

where y_2 denotes the number of ordered different pairs of \mathcal{P} that correspond to the same geometric point. We remark that the above so-called standard equations are a geometric variant of the first 3 MacWilliams identities. We will use them in the proof of Lemma 14.

3 Basic Constructions of PIR Codes

We start by considering how to lengthen a k -PIR code in order to increase the number of information bits it stores and still preserve its property as a k -PIR code. Hence, we add columns and a row to a generator matrix, where we consider the special case in which the new columns all are unit vectors with the one entry in the last row. This result is proved in the next proposition. Note that a very similar approach is also discussed in [18].

Proposition 7 *Let G be a generator matrix of an s -dimensional linear code of length n . If G can be lengthened by one row and t columns of e_{s+1} to a generator matrix G' of an $(s + 1)$ -dimensional k -PIR code, then*

- (1) G' has $n + t$ columns,
- (2) G is a k -PIR code,
- (3) for all $1 \leq i \leq s$ there exist recovery sets \mathcal{R}^i of e_i in G such that for G' each of the the recovery sets yields either e_i or $e_i + e_{s+1}$, and the latter case occurs at most t times,
- (4) there are $k - t$ recovery sets in G' for e_{s+1} , where each one of them consists of one of the last t columns and several more columns from the first n columns, while the sum of these columns in G is zero.

Proof The number of columns of G' directly follows from the construction, so that (1) holds. Let \mathcal{R}^i be the recovery sets of e_i in G' for $1 \leq i \leq s$. Over G' one of these recovery sets sums to e_i . W.l.o.g. we assume that the recovery sets are reduced so that they contain e_{s+1} at most once. If e_{s+1} is not contained in the recovery set, then it is also a recovery set for e_i in G . If e_{s+1} is contained in the recovery set, then we can remove e_{s+1} and obtain a recovery set of e_i in G that sums to $e_i + e_{s+1}$ in G' . Of course the latter case can occur at most t times since e_{s+1} is contained exactly t times as a column in G' . This gives (3) and (2). Consider the set \mathcal{R}^{s+1} of the k recovery sets of e_{s+1} in G' . At least $k - t$ recovery sets are not given by the singleton $\{e_{s+1}\}$. Since the corresponding columns in G' sum to e_{s+1} , they sum to zero in G . □

The insights of Proposition 7 can be turned into the following algorithm to generate $(s + 1)$ -dimensional k -PIR codes of length $n + t$ from s -dimensional k -PIR codes of length n .

- (1) Compute a list \mathcal{C}_1 of recovery sets $(\mathcal{R}^i)_{1 \leq i \leq s}$ of the unit vectors in G such that $|\mathcal{R}^i| = k$.
- (2) Compute a list \mathcal{C}_2 of a set \mathcal{Z} of disjoint sets whose corresponding columns in G sum to zero, where $|\mathcal{Z}| \geq k - t$.
- (3) Loop over all $r \in \mathbb{F}_2^n$ such that there exist $(\mathcal{R}^i)_{1 \leq i \leq s}$ in \mathcal{C}_1 so that for each $1 \leq i \leq s$ from the k elements in \mathcal{R}^i we have an odd number of 1s in the corresponding positions in r in at most t cases.
- (4) If there exists an element \mathcal{Z} in \mathcal{C}_2 such that from the elements in \mathcal{Z} we have an odd number of 1s in the corresponding positions in r in at least $k - t$ cases, then $\begin{pmatrix} G & 0 \\ r & 1 \dots 1 \end{pmatrix}$ is the generator matrix of an $(s + 1)$ -dimensional k -PIR code of length $n + t$.

We remark that recovery sets as well as dual codewords might be found looping over all binary vectors of length n . Collections of disjoint recovery sets \mathcal{R}^i or the disjoint zero sets \mathcal{Z} might be found by clique search. A promising heuristic is to check only binary vectors of rather small weight.

Example 8 In this example, we consider the 9-dimensional 10-PIR code of length $P(9, 10) = 28$ with generator matrix

$$\begin{pmatrix} 1000000001010100010110100101 \\ 0100000001110010110000101001 \\ 0010000000101010011010011001 \\ 0001000000011001001110001011 \\ 0000100001010011000011000111 \\ 0000010000110110001001100101 \\ 0000001000001110111000100011 \\ 0000000100000001111000011111 \\ 0000000010000000000111111111 \end{pmatrix}$$

and try $t = 3$. It comes with a unique collection of recovery sets with cardinality distribution⁵ 1^13^9 for each $1 \leq i \leq s$, which we took as our single candidate in \mathcal{C}_1 . For \mathcal{C}_2 we build up a graph with 524287 nodes and determined a clique of maximum cardinality 7. After less than 2 minutes of computation time we found the first extension $r = (00000000010001111011000000)$. For e_{10} the recovery sets in label notation are given by $\{0, 3, 8, 19\}$, $\{1, 6, 7, 16\}$, $\{2, 15, 22, 27\}$, $\{4, 11, 17, 18\}$, $\{5, 14, 24, 26\}$, $\{9, 10, 23, 25\}$, $\{12, 13, 20, 21\}$, $\{28\}$, $\{29\}$, and $\{30\}$. After less than six minutes we found 9 different extensions certifying $P(10, 10) \leq 31$ in total. So, in our example we have chosen both \mathcal{C}_1 and \mathcal{C}_2 of cardinality 1.

In general, if we apply the above algorithm as a heuristic and not for exhaustive enumeration, we do not need to find all possibilities. As mentioned above, the same applies to the possibilities for the extension row r . This rough idea leaves a lot of space for algorithmic implementations.

It was stated in [18] that the coding theoretic lower bound is tighter if, given dimension s , the number k of disjoint recovery sets is relatively small. Their formulation might be interpreted in the way that they claim that for $k = 6$ (or equivalently $k = 5$) the coding

⁵ If the recovery sets have cardinalities in $\{c_1, \dots, c_l\}$ and there are m_i recovery sets of cardinality c_i for $1 \leq i \leq l$, then we say that the recovery sets have cardinality distribution $c_1^{m_1} \dots c_l^{m_l}$. The recovery sets for e_1, \dots, e_9 are given as follows:

- e_1 : $\{0\}$, $\{1, 4, 9\}$, $\{2, 16, 17\}$, $\{3, 8, 19\}$, $\{5, 6, 13\}$, $\{7, 21, 25\}$, $\{10, 26, 27\}$, $\{11, 12, 14\}$, $\{15, 20, 23\}$, $\{18, 22, 24\}$.
- e_2 : $\{0, 4, 9\}$, $\{1\}$, $\{2, 5, 10\}$, $\{3, 23, 24\}$, $\{6, 7, 16\}$, $\{8, 13, 22\}$, $\{11, 19, 21\}$, $\{12, 25, 27\}$, $\{14, 15, 18\}$, $\{17, 20, 26\}$.
- e_3 : $\{0, 16, 17\}$, $\{1, 5, 10\}$, $\{2\}$, $\{3, 6, 12\}$, $\{4, 19, 20\}$, $\{7, 8, 23\}$, $\{9, 13, 14\}$, $\{11, 24, 25\}$, $\{15, 22, 27\}$, $\{18, 21, 26\}$.
- e_4 : $\{0, 8, 19\}$, $\{1, 23, 24\}$, $\{2, 6, 12\}$, $\{3\}$, $\{4, 7, 15\}$, $\{5, 9, 11\}$, $\{10, 16, 18\}$, $\{13, 25, 26\}$, $\{14, 20, 22\}$, $\{17, 21, 27\}$.
- e_5 : $\{0, 1, 9\}$, $\{2, 19, 20\}$, $\{3, 7, 15\}$, $\{4\}$, $\{5, 8, 21\}$, $\{6, 10, 14\}$, $\{11, 17, 18\}$, $\{12, 23, 26\}$, $\{13, 24, 27\}$, $\{16, 22, 25\}$.
- e_6 : $\{0, 6, 13\}$, $\{1, 2, 10\}$, $\{3, 9, 11\}$, $\{4, 8, 21\}$, $\{5\}$, $\{7, 12, 18\}$, $\{14, 24, 26\}$, $\{15, 19, 25\}$, $\{16, 20, 27\}$, $\{17, 22, 23\}$.
- e_7 : $\{0, 5, 13\}$, $\{1, 7, 16\}$, $\{2, 3, 12\}$, $\{4, 10, 14\}$, $\{6\}$, $\{8, 15, 26\}$, $\{9, 21, 22\}$, $\{11, 23, 27\}$, $\{17, 19, 24\}$, $\{18, 20, 25\}$.
- e_8 : $\{0, 21, 25\}$, $\{1, 6, 16\}$, $\{2, 8, 23\}$, $\{3, 4, 15\}$, $\{5, 12, 18\}$, $\{7\}$, $\{9, 20, 24\}$, $\{10, 13, 17\}$, $\{11, 22, 26\}$, $\{14, 19, 27\}$.
- e_9 : $\{0, 3, 19\}$, $\{1, 13, 22\}$, $\{2, 7, 23\}$, $\{4, 5, 21\}$, $\{6, 15, 26\}$, $\{8\}$, $\{9, 18, 27\}$, $\{10, 11, 20\}$, $\{12, 16, 24\}$, $\{14, 17, 25\}$.

theoretic bound is always at least as tight as Inequality (3).⁶ However, this is not the case. An example is given by $s = 92$ and $k = 5$, where $N(92, 5) = 106^7$ but $P(92, 5) \geq 107$ due to Inequality (3). Also for $k > 6$ there are such examples, however, they require rather large values of s . So, the situation should be as follows: If $k > 4$ and the dimension s is not too big with respect to k , then the coding theoretic lower bound is superior. If the dimension gets huge, then Inequality (3) is tighter. For $k = 3$ or $k = 4$, see Footnote 3.

Using even more geometric terms, we can formulate a parametric construction. The rough idea is as follows. We start with an s -dimensional simplex code, represented by a set of points \mathcal{P} , and the recovery sets stated in Footnote 4. A line in \mathbb{F}_2^s is a set of three collinear points, i.e., three non-zero vectors a, b, c in \mathbb{F}_2^s with $a + b + c = \mathbf{0}$. We iteratively remove the three points from a line from \mathcal{P} and modify the list \mathcal{R}^i of recovery sets accordingly. A partial line spread in \mathbb{F}_2^s is a set of lines that do not have a non-zero vector in common. The maximum possible cardinality of a line spread is $\frac{2^s-1}{3}$ if s is even and $\frac{2^s-5}{3}$ if s is odd, see e.g. [3].

Proposition 9 For every integer $s \geq 3$ and every integer $0 \leq \lambda \leq \frac{2^{s-1}-3-2 \cdot (-1)^s}{3}$ we have $P(s, 2^{s-1} - 2\lambda) \leq 2^s - 1 - 3\lambda$.

Proof Let \mathcal{P} be the set of $2^s - 1$ points in \mathbb{F}_2^s and \mathcal{R}^i the corresponding lists of recovery sets for $1 \leq i \leq s$ of the s -dimensional simplex code, see Footnote 4. We will remove 3λ points from \mathcal{P} and modify the set \mathcal{R}^i accordingly. To this end, let H be a hyperplane of \mathbb{F}_2^s not containing any unit vector and \mathcal{L} be a maximal partial line spread of H of cardinality $\frac{2^{s-1}-1}{3}$ if s is odd and of cardinality $\frac{2^{s-1}-5}{3}$ if s is even. Note that the upper bound for λ equals this cardinality. For an arbitrary subset of \mathcal{L} of cardinality λ we remove the corresponding 3λ points from \mathcal{P} to get the set of points of our final PIR code. We also have to adjust the lists \mathcal{R}^i . Since H does not contain any unit vector, all elements of the set \mathcal{R}^i have cardinality two in the beginning and no recovery set of cardinality two is completely contained in a line from \mathcal{L} . So, consider one line that is removed and assume that its points are given by $\{a, b, c\}$. For a fixed but arbitrary $1 \leq i \leq s$ the recovery sets for e_i that contain either $a, b, \text{ or } c$ are given by $\{a, a + e_i\}, \{b, b + e_i\}, \text{ and } \{c, c + e_i\}$. Those three recovery sets are destroyed by our operation of removing $\{a, b, c\}$, but we can additionally add the recovery set $\{a + e_i, b + e_i, c + e_i\}$, noting that

$$\begin{aligned} &(a + e_i) + (b + e_i) + (c + e_i) \\ &= (a + b + c) + (e_i + e_i + e_i) = 0 + e_i = e_i. \end{aligned}$$

It remains to check that no point x is used in two recovery sets of cardinality three for the same e_i and that we do not remove a point y that is contained in a constructed recovery set of cardinality three. If x is contained in two recovery sets of cardinality three for e_i , then $x + e_i$ is contained in two lines of \mathcal{L} , which contradicts the disjointness. If y is removed, then $y \in H$. If, additionally, y is contained in a recovery set of cardinality three for e_i , then $y + e_i$ is removed, so that $y + e_i \in H$. Thus, $e_i \in H$, which is a contradiction. \square

Note that Proposition 9 improves the best known upper bounds for $P(5, 10), P(5, 12), \text{ and } P(5, 14)$ compared to [11, Table I]. For $P(5, 12)$ we start from the 5-dimensional simplex

⁶ "It can be easily verified that in general $N(k, t) \leq L_P(k, t)$ for small values of $t > 4$. In fact, we will show in Section V that $N(k, t)$ is a tighter lower bound on $N_P(k, t)$ than $L_P(k, t)$ for $r = 6$."

⁷ The value $N(92, 5) = 106$ is taken from <http://mint.sbg.ac.at>.

code with generator matrix

$$\begin{pmatrix} 00000000000000001111111111111111 \\ 0000000011111111100000000011111111 \\ 000111110000111110000111110000111111 \\ 01100101001100101001100101001100101 \\ 10101011010101011010101011010101011 \end{pmatrix}.$$

As hyperplane H we can choose the set of 15 non-zero vectors in \mathbb{F}_2^5 that are perpendicular to the all-one vector $(11111)^\top$:

$$\begin{matrix} 0000000111111111 \\ 0001111000011111 \\ 0110011001100111. \\ 1100101010101010 \\ 1011001100101101 \end{matrix}$$

Two disjoint lines in H are given by

$$\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\} \text{ and } \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\},$$

so that we obtain a 5-dimensional 12-PIR code with length 25 and generator matrix

$$\begin{pmatrix} 00000000000000001111111111111111 \\ 00000111111111000000001111111111 \\ 001110001111100011111000111111 \\ 01001011001010110010101100101 \\ 10011001010110010101110101011 \end{pmatrix}.$$

Corollary 10 For every integer $s \geq 2$ we have $P(s, 2^{s-1} - 2) = 2^s - 4$

Proof Since the result is trivial for $s = 2$ we assume $s \geq 3$ and consider the lower bound

$$P(s, 2^{s-1} - 2) \geq \left\lceil (2^{s-1} - 2) \cdot \frac{2^s - 1}{2^{s-1}} \right\rceil = 2^s - 4.$$

This lower bound is attained by the construction from Proposition 9. □

4 Integer Linear Programming Formulations

In this section we present an integer linear programming (ILP) formulation for the exact determination of $P(s, k)$. Given the dimension s , we set $\mathcal{X} = \mathbb{F}_2^s \setminus \{\mathbf{0}\}$. The generator matrix is sufficiently characterized if we know for each element $j \in \mathcal{X}$ the integer-valued multiplicity x_j of that column. By \mathcal{Y}^i we denote the set of all minimal recovery sets for e_i . For each $j \in \mathcal{Y}^i$ we denote by the integer y_j^i the number of times recovery set j for e_i is used. With this, the value of $P(s, k)$ is given by

$$\min \sum_{j \in \mathcal{X}} x_j \tag{8}$$

$$\sum_{h \in \mathcal{Y}^i : j \in h} y_h^i \leq x_j \quad \forall j \in \mathcal{X}, \forall 1 \leq i \leq s \tag{9}$$

$$\sum_{j \in \mathcal{X} \setminus H} x_j \geq k \quad \forall H \leq \mathbb{F}_2^s, \dim(H) = s - 1 \tag{10}$$

$$\sum_{h \in \mathcal{Y}^i} y_h^i \geq k \quad \forall 1 \leq i \leq s \tag{11}$$

$$x_j \in \mathbb{N} \quad \forall j \in \mathcal{X} \tag{12}$$

$$y_h^i \in \mathbb{N} \quad \forall 1 \leq i \leq s, \forall h \in \mathcal{Y}^i \tag{13}$$

Inequality (9) guarantees that the column multiplicities are sufficiently large for the chosen recovery sets. Inequality (11) ensures that there are at least k disjoint recovery sets for each e_i .

Inequality (10) implements Lemma 6.

In principle, Inequality (10) is not necessary, but in practice it usually speeds up the solution process. Additional lower and upper bounds on x_j can be deduced from Proposition 20, Lemma 22, respectively. (Tighter bounds can be obtained from more sophisticated coding theoretic arguments, see

Section 5.)

The problem with that ILP formulation is that it quickly gets too huge to be solved exactly. More precisely, the number of constraints is in $O(s2^s)$ and the number of variables is in $O(s2^{2s})$. So, it is applicable for rather small parameters s and k only, where the size of k plays almost no role.

By imposing further restrictions we can use the above ILP formulation as a heuristic to find good codes that eventually attain the known lower bounds or improve known constructions from the literature. The restriction to systematic PIR codes can be enforced by $x_{e_i} \geq 1$ for all $1 \leq i \leq s$ and the restriction to projective PIR codes can be implemented by $x_j \in \{0, 1\}$ for all $j \in \mathcal{X}$. If the dimension s is not too large, then $|\mathcal{X}| = 2^s - 1$, i.e., the number of x variables is still manageable. In order to prevent the combinatorial explosion of $|\mathcal{Y}^i|$ we can restrict ourselves to recovery sets of cardinality at most λ by modifying the definition of \mathcal{Y}^i accordingly. In our numerical experiments we mostly chose $\lambda = 3$ (and $\lambda = 4$ in some very small cases). The intuition between this heuristic is as follows. We know that the simplex code is optimal and uses recovery sets of cardinalities 1 and 2 only. For large values of k the coding theoretic lower bound is usually not too far away from the fractional simplex code, i.e., the mentioned lower bound $P(s, k) \geq N(s, k) \geq \frac{2^s - 1}{2^s - 1} \cdot k$. So, for a good PIR code not too many recovery sets of cardinality larger than 2 can occur. There is some hope that recovery sets of cardinality at most 3 are sufficient provided k is large enough. The ILP constructions in Table 1 support this hope. For smaller values of k it seems that larger recovery sets are necessary.

Another way to decrease the computational complexity is to prescribe a subgroup of the final automorphism group of the code. In our context, automorphisms are permutations of the columns and rows of the generator matrix, c.f. [13] where transitive automorphism groups are considered. Counting point multiplicities for the columns, as above, this leaves row permutations only. So, for some subgroup $H \leq S_s$ of the symmetric group on s elements we can require $x_j = x_{\pi j}$ and $y_j^i = y_{\pi j}^i$ for every $\pi \in H$, $1 \leq i \leq j$, and $j \in \mathcal{X}$ or $j \in \mathcal{Y}^i$, respectively. This reduces the number of variables as well as the number of constraints, since several of them become identical, roughly by a factor of the group size $|H|$. If the corresponding substitutions and removals of identical constraints are performed directly,

then this is also known under the name Kramer-Mesner approach. As an example we state that prescribing a cyclic \mathbb{Z}_6 allowed us to construct an example for $P(6, 16) \leq 36$ improving the previous bound $P(6, 16) \leq 39$ [28] and prescribing a cyclic \mathbb{Z}_3 ⁸ allowed us to construct an example for $P(7, 16) \leq 39$ improving the previous bound $P(7, 16) \leq 43$ [28]. Lastly, we list some of the generator matrices found by the ILP approach for dimension $s \geq 2$ and $k \geq 4$ in Appendix A.

With respect to lower bounds, we remark that it is possible to modify the initial ILP formulation by restricting the possible sizes of recovery sets to at most λ while still obtaining a lower bound that is valid without this assumption. Given a fix value of λ we cannot require Inequality (11) any more, since we ignore recovery sets that have a cardinality strictly larger than λ . If we set $n = \sum_{j \in \mathcal{X}} x_j$, then we can replace Inequality (11) with the following relaxation:

$$n + \sum_{j \in \mathcal{Y}^i} (\lambda + 1 - |j|) \cdot y_j^i \geq (\lambda + 1) \cdot k \quad \forall 1 \leq i \leq s, \tag{14}$$

where $|j|$ denotes the size of the recovery set. The idea is simple: $k - \sum_{j \in \mathcal{Y}^i} y_j^i$ recovery sets for e_i have to be of cardinality at least $\lambda + 1$ and the total size of all recovery sets for e_i is at most n . So, *choosing* recovery sets of large size has no consequences for the x_j directly but imposes lower bounds on n , which is a relaxation of the original inequality. In the other direction, Inequality (10) can be tightened:

$$\begin{aligned} \sum_{j \in \mathcal{X} \setminus H} x_j &\geq k + \sum_{j \in \mathcal{Y}^i} (|j \cap \mathbb{F}_2^s \setminus H| - 1) y_j^i \\ \forall 1 \leq i \leq s \forall H \leq \mathbb{F}_2^s, \dim(H) = s - 1, e_i \notin H, \end{aligned} \tag{15}$$

where $j \cap \mathbb{F}_2^s \setminus H$ denotes the number of elements of the recovery set $j \in \mathcal{Y}^i$ that are not contained in H . The argument for the hyperplane conditions of Inequality (10), see Lemma 6, was that vectors in H cannot build a recovery set for e_i on their own for at least one $1 \leq i \leq s$, so that at least one column outside H is needed for each recovery set. Now, if we fix i and we know that some recovery sets use more than one column outside of H than the total number of columns outside of H increases, which gives Inequality (15).

We remark that for $\lambda \leq 2$ Inequality (15) is the same as Inequality (10). Assume the contrary and suppose that for a given hyperplane H of \mathbb{F}_2^s and a given unit vector e_i both points of the two-element recovery set $\{a, b\}$ for e_i are not contained in H . Since the line $\{a, b, a + b + e_i\}$ intersects H in a point, we have $e_i \in H$, which is a contradiction.

We call the ILP (8)-(10), (12)-(15) the lower bound ILP for a given value of λ . We remark that the lower bound ILP for $\lambda = 3$ increases the coding theoretic lower bound by 1 in the cases $(s, k) \in \{(4, 3), (4, 4), (4, 12), (5, 10), (5, 12), (6, 8), (6, 12)\}$, by 2 for $(s, k) \in \{(5, 8), (6, 14)\}$, and by 4 for $(s, k) = (6, 16)$, cf. Table 1. We remark that the problem for $(s, k) = (6, 16)$ was too tight to be solved directly. Here we applied symmetry breaking techniques and additional inequalities. More concretely, we started by adding $x_{e_1} \geq x_{e_2} \geq \dots \geq x_{e_6}$, $n \leq 35$ and maximizing e_1 . After an upper bound of one was verified we stopped and concluded the additional inequalities $x_{e_i} \leq 1$ for all $1 \leq i \leq 6$. Similarly, we tried $x_{1-e_1} \geq x_{1-e_2} \geq \dots \geq x_{1-e_6}$, $n \leq 35$ and maximized x_{1-e_1} . After 14 028 seconds and 4 129 360 branch&bound (B&B) nodes an upper bound of 1 was verified, so that $x_{1-e_i} \leq 1$ are valid inequalities for all $1 \leq i \leq 6$ (assuming $n \leq 35$). With these additional ≤ 1 -inequalities we started a last round of symmetry breaking: We introduced the integer variables s_i counting the sum of those x_p where point p has a one in the i th coordinate. By symmetry we can assume

⁸ Note that for $s = 7$ there are two possible cycle types of the cyclic group \mathbb{Z}_3 in S_7 up to conjugation.

$s_1 \geq s_2 \geq \dots \geq s_6$. Maximizing s_1 with the additional assumption $n \leq 35$ was solved after 102 967 seconds and 24 767 382 B&B nodes to be infeasible, so that $P(6, 16) = 36$. We applied the same technique to computationally verify $P(6, 14) \geq 32$ in 21 566 seconds and 12 701 361 B&B nodes.

Another interesting instance is:

Proposition 11 *Each systematic 6-dimensional 8-PIR code of length n satisfies $n \geq 20$, while $P(6, 8) = 19$.*

Proof We apply the lower bound ILP with $\lambda = 3$ while additionally prescribing the use of the vectors from an s -dimensional unit matrix. After less than 900 seconds and 125603 B&B nodes a solution with $n = 20$ was proven to be optimal. □

5 Lower Bounds and the Dual Minimum Distance/Dual Code(s)

In this section we want to use coding theoretic methods to provide lower bounds for $P(s, k)$. To this end let C be an $[n, s]$ code. The corresponding dual code C^\perp is the $[n, n - s]$ code whose codewords are those that are perpendicular to all codewords in C . By d^\perp we denote the minimum distance of C^\perp , which is also called the dual minimum distance.

Lemma 12 *Let C be a linear $[n, s, d]$ code with minimum dual distance d^\perp and generator matrix G .*

- (a) *If R and R' are two different recovery sets for the same symbol i in G , then $|R| + |R'| \geq d^\perp$.*
- (b) *If G is a k -PIR generator matrix with $k \geq 2$, then $n \geq k\lceil d^\perp/2 \rceil - 1$ if d^\perp is odd and $n \geq k\lceil d^\perp/2 \rceil$ if d^\perp is even.*
- (c) *If G is a k -PIR generator matrix that contains a unit vector e_i , then $n \geq 1 + (k - 1)(d^\perp - 1)$.*

Proof Since $\sum_{j \in R} G^j = e_i = \sum_{j \in R'} G^j$, we have $\sum_{j \in (R \cup R') \setminus (R \cap R')} G^j = 0$, i.e., $(R \cup R') \setminus (R \cap R')$ is the support of a dual codeword, which gives part (a). Next, we consider an arbitrary unit vector e_i and let m denote the cardinality of the smallest recovery set for e_i . From (a) we conclude that every other recovery set has cardinality at least $d^\perp - m$, so that $n \geq (k - 1)(d^\perp - m) + m$. The special case $m = 1$ corresponds to (c). For part (b) we can argue as follows. If $m \geq \lceil d^\perp/2 \rceil$, then $n \geq k\lceil d^\perp/2 \rceil$, so that we assume $m \leq \lceil d^\perp \rceil - 1$ and conclude

$$n \geq (k - 1)(d^\perp - m) + m \geq (k - 2)\lceil d^\perp/2 \rceil + d^\perp. \quad \square$$

Proposition 13 *For each integer $s \geq 4$ we have $P(s, 2^{s-2}) \geq 2^{s-1} + 1$.*

Proof It is well known that $N(s, 2^{s-2}) = 2^{s-1}$ with the unique solution being the first order Reed-Muller code, i.e., in geometric terms, all points of \mathbb{F}_2^s except those in a distinguished hyperplane, see Lemma 14 for a short self-contained proof. As no multiple points or lines (sets of three collinear points) are contained, the dual minimal distance d^\perp is at least 4 (indeed it is 4). Let G be a generator matrix that is a 2^{s-2} -PIR code. If G contains a unit vector, then Lemma (12).(c) gives $n \geq 1 + (2^{s-2} - 1) \cdot 3$, which is a contradiction for $s \geq 4$. Thus, G does not contain any unit vector and every recovery set has cardinality at least 2. Since $n = 2k$ every recovery set has cardinality exactly 2.

In order to obtain a contradiction we now prove the following statement by induction on $1 \leq j \leq s - 1$. For each $1 \leq j \leq s - 1$ there exist vectors $x_1, \dots, x_\ell \in \mathbb{F}_2^s$, where $\ell = 2^{s-1-j}$, such that the columns of G are given by $\{x_h + \langle e_1, \dots, e_j \rangle \mid 1 \leq h \leq \ell\}$, where we slightly abuse notation. By $x_h + \langle e_1, \dots, e_j \rangle$ we abbreviate the list of 2^j vectors contained in the affine \mathbb{F}_2 -vector space $x_h + \langle e_1, \dots, e_j \rangle$.

For the induction start we remark that the 2^{s-1} columns are partitioned into pairs $\{x_h, x_h + e_1\}$ corresponding to the recovery sets of e_1 . For the induction step we assume that the columns of G are partitioned into $\ell = 2^{s-1-j}$ sets, which we call blocks, of the form $x_h + \langle e_1, \dots, e_j \rangle$. Now we are considering the recovery sets of e_{j+1} . Let $R = \{a, b\}$ be a recovery set of e_{j+1} that was not considered before. Note that a and b have to be contained in different blocks since $a + b = e_{j+1}$. W.l.o.g. let a be contained in the first block $x_1 + \langle e_1, \dots, e_j \rangle$ and b in the second block $x_2 + \langle e_1, \dots, e_j \rangle$, so that we can reparameterize to $a + \langle e_1, \dots, e_j \rangle$ and $b + \langle e_1, \dots, e_j \rangle$. Since $a + b = e_{j+1}$ the union of the two blocks can be described by $\mathcal{B} = a + \langle e_1, \dots, e_j, e_{j+1} \rangle$. Note that this new block \mathcal{B} contains 2^j recovery sets for e_{j+1} of cardinality 2. In principle those recovery sets do not need to coincide with those from which we started. However, we can perform the following swaps. Let $\{b_1, c_1\}$ be a recovery set for e_{j+1} that has not been considered so far with $b_1 \in \mathcal{B}$ and $c_1 \notin \mathcal{B}$. Let $b_2 \in \mathcal{B}$ with $b_1 + b_2 = e_{j+1}$ and $c_2 \notin \mathcal{B}$ with $c_2 + b_2 = e_{j+1}$. Instead of the recovery pairs $\{b_1, c_1\}$ and $\{b_2, c_2\}$ we swap to the recovery pairs $\{b_1, b_2\} \subseteq \mathcal{B}$ and $\{c_1, c_2\}$. Thus, we can assume that all nodes of \mathcal{B} pair within \mathcal{B} . Going on with another unconsidered recovery pair gives us a new block each time so that the induction step is proven.

For e_s we use the structural information that the columns of G can be described as $x + \langle e_1, \dots, e_{s-1} \rangle$ for some vector $x \in \mathbb{F}_2^s$. Thus there can be no recovery set of cardinality two for e_s .

Since the first order Reed-Muller code is excluded, we have $P(s, 2^{s-2}) \geq N(s, 2^{s-2}) + 1 = 2^{s-1} + 1$. □

We remark that the uniqueness of the first order Reed-Muller code is not needed in the above proof. It is sufficient to have the information that any length optimal code with dimension s and minimum distance 2^{s-2} satisfies $d^\perp \geq 4$, which can be concluded from the MacWilliams equations.

Another application of Lemma 12 is to use the uniqueness of the binary extended Golay code with parameters $[24, 12, 8]$, see [23]⁹. Since the code is self-dual, we have $d^\perp = 8$ so that part (b) implies that any generator matrix of the binary extended Golay code cannot be a 7-PIR code. Since $N(12, 8) = 24$, this implies $P(12, 8) \geq 25$.

As a further relation between the minimum dual distance d^\perp and PIR codes we note that the lower bound (3) was proven in [30] using the dual code and especially the minimum dual distance.

Now let us consider linear codes that generalize the parameters of a first order Reed-Muller code.

Lemma 14 *Let $s \geq 1$ and $\ell \geq 0$ be integers and C be a linear $[2^{s-1} + \ell(2^s - 1), s, (2\ell + 1)2^{s-2}]$ code. If \mathcal{P} denotes the corresponding multiset of points, then the multiplicity of every point in \mathbb{F}_2^s with respect to \mathcal{P} is either ℓ or $\ell + 1$. Moreover, the $2^{s-1} - 1$ points with multiplicity ℓ form a hyperplane in \mathbb{F}_2^s .*

⁹ The code is even unique within the class of non-linear codes, see [9].

Proof Assume that there exists a point p with multiplicity at least $\ell + 2$. Shortening gives an

$$[\leq (2\ell + 1) \cdot 2^{s-1} - 2\ell - 2, s - 1, (2\ell + 1)2^{s-2}]$$

code, which is a contradiction since each $[n', s - 1, (2\ell + 1)2^{s-2}]$ code satisfies

$$n' \geq (2\ell - 1) \cdot (2^{s-1} - 1) = (2\ell + 1) \cdot 2^{s-1} - 2\ell - 1,$$

see Inequality (2).

Now consider the complementary multiset of points \mathcal{P}' where the multiplicity of each point of \mathbb{F}_2^s is given by $\ell + 1$ minus its original multiplicity with respect to \mathcal{P} . Counting points gives that $|\mathcal{P}'| = 2^{s-1} - 1$. Let H be an arbitrary hyperplane of \mathbb{F}_2^s . Due to Inequality (4) in Lemma 6 we have

$$|(\mathcal{P}' \cap H)| \geq 2^{s-2} - 1$$

for every hyperplane. Now we are using linear combinations of the left hand and right hand sides of the standard equations. Using the abbreviation $x = 2^{s-2} - 1, x(2x + 1)$ times Equation (5) minus $3x$ times Equation (6) plus Equation (7) gives

$$\sum_{i \geq 0} (2x + 1 - i)(x - i)h_i = (x + 1)y_2.$$

Due to Inequality (4) we have $h_i = 0$ for $i < x$ and since the number of points is $2x + 1$, we have $h_i = 0$ for $i > 2x + 1$. The coefficient $(2x + 1 - i)(x - i)$ of h_i is zero for $i \in \{x, 2x + 1\}$ and strictly negative for all $x < i < 2x + 1$. Since $h_i \geq 0$ for all integer i the left hand side is at most zero. From $x \geq 0$ and $y_2 \geq 0$ we conclude that the right hand side is at least zero, so that both side have to be equal to zero. This directly implies $y_2 = 0$ and $h_i = 0$ for all $x < i < 2x + 1$. From Equation (5) and Equation (6) we then conclude that $h_x = 2^{s-2}$ and $h_{2x+1} = 1$. $y_2 = 0$ tells us that the point multiplicity with respect to \mathcal{P}' is at most 1, so that the point multiplicity with respect to \mathcal{P} is at least ℓ . From $h_{2x+1} = 1$ we read of that there is exactly one hyperplane H whose $2x + 1 = 2^{s-1} - 1$ points form the set \mathcal{P}' , so that the stated result follows. □

We remark that a more complicated proof has been given for example in [6]. However, the result should be well-known for several decades.

Lemma 15 *Let C be an s -dimensional binary k -PIR code of length n that contains every non-zero vector of \mathbb{F}_2^s at least once as a column of a generator matrix. Then, $n \geq P(s, k - 2^{s-2}) + 2^{s-1} - 1$.*

Proof Let \mathcal{R}^i be corresponding recovery sets. We will now show that we can modify the recovery sets so that they contain the recovery sets of the s -dimensional simplex code as a subset. First of all, we assume that all recovery sets in \mathcal{R}^i are minimal. Especially, we have that $\{e_i\}$ is contained in \mathcal{R}^i . Due to symmetry we only consider the modification of \mathcal{R}^1 .

For every vector $x \in \mathbb{F}_2^s \setminus \{0\}$ with first coordinate equal to zero we have the recovery set $\{x, x + e_1\}$ in the simplex code. If that recovery set is contained in \mathcal{R}^1 , nothing has to be done. Otherwise x is contained in a recovery set A with $|A| \geq 3$ and $x + e_1$ is contained in a recovery set B with $|B| \geq 3$. We replace the recovery sets A and B by $\{x, x + e_1\}$ and $A \cup B \setminus \{x, x + e_1\}$ (considered as a multiset union or with removed duplicates). □

From Proposition 13, Lemma 14 and Lemma 15 we iteratively conclude:

Corollary 16 For each integer $s \geq 4$ and each integer $\ell \geq 0$ we have $P(s, 2^{s-2} + \ell 2^{s-1}) \geq \ell(2^s - 1) + 2^{s-1} + 1$.

An example is $P(4, 12) \geq 24$, which can indeed be attained and improves the coding theoretic lower bound $N(4, 12) = 23$ by one. We note that we only need the information that every non-zero point of \mathbb{F}_2^s is taken at least once, i.e., $y_2 = 0$ for the complementary multiset of points.

Another example, which is a bit more involved, is $P(5, 8) \geq 18$. The coding theoretic lower bound $N(5, 8) = 16$ is improved by two. To this end, we say that a code has weight distribution $w_1^{m_1} w_2^{m_2} \dots$ if there are m_1 codewords of weight w_1 , m_2 codewords of weight w_2 , and so on.

Lemma 17 If $P(5, 8) = 17$, then the corresponding PIR code is projective and has weight distribution $0^1 8^{14} 9^{16} 16^1$.

Proof Let P be a column with multiplicity m . Shortening then gives a $[17 - m, 4, 8]$ code C' , which implies $m \leq 2$. So, we assume $m = 2$ and note that the unique $[15, 4, 8]$ code is the 4-dimensional simplex code. I.e., the columns of a generator matrix G' of C' consist of all non-zero vectors of \mathbb{F}_2^4 . Now consider any lengthened $[17, 5, 8]$ code. The two new columns can be contained in at most two different recovery sets for e_5 . Recovery sets for e_5 that consist solely of some of the first 15 columns of G' have cardinality at least three, since C' has dual minimum distance 3, i.e., no two columns of G' sum to zero. Thus, $2 \cdot 1 + 6 \cdot 3 > 17$ gives a contradiction, so that $m = 1$ and the code has to be projective.

Finally, we note that there are exactly four $[17, 5, \geq 8]$ codes. Only one of these is projective and has the stated weight distribution.¹⁰ □

The unique code determined in Lemma 17 has a dual weight distribution starting with

$$0^1 3^8 4^{140} 5^{112} 6^{448} \dots$$

and can be generated by

$$\begin{pmatrix} 11111110000010000 \\ 11110001110001000 \\ 11001101101000100 \\ 00111101100100010 \\ 10101011011100001 \end{pmatrix}.$$

The weight distribution of dual codes can be used even more directly than in the proof of Lemma 17. Assume that we have an s -dimension k -PIR code of length n with generator matrix G . Let G' denote the matrix that arises if we remove the i th row of G . The recovery sets of cardinality w for e_i in G correspond to dual codewords in G' of weight exactly w . Obviously, this is not a bijection, since we completely ignore the entries in the i th row of G .

Lemma 18 Let G is the generator matrix of an s -dimensional k -PIR code of length n . If the cardinality vector of \mathcal{R}^i , where $1 \leq i \leq s$ is arbitrary, is given by $1^{m_1} 2^{m_2} 3^{m_3} \dots$ (clearly $\sum_j m_j = k$), then there exists a matrix G' that is the generator matrix of an $(s - 1)$ -dimensional k -PIR code of length $n - m_1$ such that there exist $k - m_1$ disjoint dual codewords with weight distribution $2^{m_2} 3^{m_3} \dots$.

¹⁰ All exhaustive lists of binary linear codes have been enumerated with the software package LinCode, see [16].

Proof Apply expurgation, i.e., remove the i th row from G . □

Theorem 19

$$P(5, 8) = 18$$

Proof Due to Corollary 16 we only have to consider length $n = 17$. We apply Lemma 18 and enumerate the $[17, 4, \geq 8]$ codes C_i . There are exactly 23 of them. However, we can use more information of a putative 5-dimensional 8-PIR code of length 17. The possible cardinality vectors of one list \mathcal{R}^i are

$$1^1 2^5 3^2, 1^1 2^6 3^1, 1^1 2^6 4^1, 1^1 2^7, 2^7 3^1,$$

i.e., in any case we have at least five recovery sets of cardinality 2. For the codes C_i this translates to the requirement that in a generator matrix there have to be at least five disjoint pairs of identical columns and at least 7 disjoint pairs of identical columns if the effective length is 17. This leaves the following four codes with generator matrices

$$\begin{pmatrix} 1111111000001000 \\ 11110001110000100 \\ 11001101101000010 \\ 0011110110010001 \end{pmatrix}, \begin{pmatrix} 11111110000001000 \\ 11110001110000100 \\ 11001101101100010 \\ 00111101100010001 \end{pmatrix}, \\ \begin{pmatrix} 11111110000001000 \\ 11110001110000100 \\ 11001101101100010 \\ 11110000001110001 \end{pmatrix}, \begin{pmatrix} 11111110000001000 \\ 11110001110000100 \\ 11110000001110010 \\ 10001101101100001 \end{pmatrix},$$

see Footnote 10.

The last one contains a column with multiplicity 3. Since by adding an additional row to the generator matrix the multiplicity of each point can decrease by a factor of at most 2, this contradicts Lemma 17.

The first one is a doubled Reed-Muller code¹¹ with dual weight distribution $0^1 2^8 4^{252} 6^{952} \dots$. Of course the code itself is a 8-PIR code. No recovery set of cardinality one can be used, since there is no dual codeword of weight 3 and at least one recovery set of cardinality 2 has to be used for each e_i . Thus, every cardinality set has exactly cardinality 2. So, everything could be partitioned into two halves and we would obtain two 4-dimensional 4-PIR codes of length 8 each, which do not exist.

The second code has weight distribution $0^1 8^7 9^4 11^4$, so that it clearly cannot be augmented to a code with weight distribution $0^1 8^{14} 9^{16} 16^1$ due to the codeword of weight 11.

Lastly, there remains the third code with weight distribution $0^1 8^6 9^8 16^1$. Thus, a 5-dimensional 8-PIR code has a generator matrix without any unit vector, since expurgation would otherwise give a 4-dimensional code of length strictly less than 17. Hence, the cardinality distribution for every \mathcal{R}^i is $2^7 3^1$ and all rows of the generator matrix have weight of exactly 8. (In a recovery set of cardinality 2 for e_i there is exactly one 1 in coordinate i and in a recovery set of cardinality 3 there are either 1 or 3 ones in coordinate i . Since there is no codeword of weight 10 in the code the stated observation follows.) So, the weight of any row of the generator matrix is divisible by 8, the sum of the weights of any two different rows is divisible by 4, and the sum of the weights of any three different rows is divisible by 2. Thus, the number of codewords of weight 9 is at most $\binom{2}{4} + \binom{2}{5} = 6 < 16$, which is a contradiction. □

¹¹ We refer to a doubled Reed-Muller code as the code generate by concatenating a generator of a Reed-Muller code twice with itself.

6 Bounds and Exact Values of PIR Codes

Lemma 15 has another important consequence.

Proposition 20 *For each positive integer $\ell \geq P(s, k) - 2k$ we have*

$$P(s, k + 2^{s-1} \cdot \ell) = P(s, k + 2^{s-1} \cdot (\ell - 1)) + 2^s - 1.$$

Proof Since $P(s, 2^{s-1}) \leq 2^s - 1$, we obviously have $P(s, k + 2^{s-1} \cdot \ell) \leq P(s, k + 2^{s-1} \cdot (\ell - 1)) + 2^s - 1$. Now let G be a generator matrix of a matching PIR code attaining length $P(s, k + 2^{s-1} \cdot \ell)$. If every non-zero vector of \mathbb{F}_2^s occurs as a column of G , then Lemma 15 gives $P(s, k + 2^{s-1} \cdot \ell) \geq P(s, k + 2^{s-1} \cdot (\ell - 1)) + 2^s - 1$. Thus, it remains to assume the existence of a non-zero point $P \in \mathbb{F}_2^s$ with multiplicity zero. By x_j we denote the number of occurrences of vector $j \in \mathbb{F}_2^s \setminus \{0\}$ as column vector of G . From Inequality (4) we conclude $\sum_{j \notin H} x_j \geq k'$ for every hyperplane H of \mathbb{F}_2^s , where $k' = k + 2^{s-1} \cdot \ell$. Summing over all 2^{s-1} hyperplanes that do not contain P gives

$$\sum_{H \subseteq \mathbb{F}_2^s : \dim(H)=s-1, P \notin H} \sum_{j \notin H} x_j \geq 2^{s-1} \cdot k'.$$

The coefficient of x_P on the left hand side is 2^{s-1} and for any other non-zero point $Q \neq P$ the coefficient of x_Q on the left hand side is given by

$$\frac{2^{s-1} \cdot (2^{s-1} - 1)}{2^s - 2} = 2^{s-2},$$

so that $n \geq 2k'$, where $n = \sum_j x_j$ (using $x_P = 0$). Thus,

$$n \geq 2k' = (2k + l) + (2^s - 1) \cdot \ell.$$

Since $P(s, k + 2^{s-1} \cdot \ell) \leq P(s, k) + (2^s - 1) \cdot \ell$, the stated result follows. □

As an example we use $P(4, 4) = 9$ to conclude $P(4, 12) = 24$, or more generally $P(4, 4 + 8\ell) = 9 + 15\ell$ for every integer $\ell \geq 0$, c.f. Proposition 23. We remark that the condition $\ell \geq P(s, k) - 2k$ in Proposition 20 can be replaced by $\ell \geq \mu - 2k$, where μ is some arbitrary upper bound for $P(s, k)$.

Corollary 21 *For every fixed integer $s \geq 1$ the determination of the function $P(s, \cdot)$ is a finite problem.*

Proof Choose an integer ℓ such that $\ell \geq P(s, k) - 2k$ for all $1 \leq k \leq 2^{s-1}$. Due to Proposition 20, the values $P(s, k)$ for $k \leq \ell \cdot 2^{s-1}$ completely determine the function $P(s, \cdot)$. □

While the above example assures that $P(4, 4 + 8\ell) = 9 + 15\ell$, the value $P(s, k)$ does not need to tend to the coding theoretic lower bound $N(s, k)$, which exactly approaches the Griesmer bound, for sufficiently large values of k . Hence, Corollary 21 generalizes the result from [2] for the Griesmer bound to PIR codes.

In a similar style, as in the proof of Proposition 20, we can state an easy to evaluate lower bound on the code size if some point has a large multiplicity.

Lemma 22 *Let C be an $[n, s, d]$ code where one column of a generator matrix has multiplicity m . Then, $n \geq \frac{2^{s-1}-1}{2^{s-2}} \cdot d + m$.*

Proof By x_j we denote the number of columns of a given generator matrix G of C that are equal to a non-zero vector $j \in \mathbb{F}_2^s$. Let P be non-zero vector with multiplicity m . Summing Inequality (4) over all hyperplanes that contain P , we obtain

$$\sum_{H \leq \mathbb{F}_2^s : \dim(H)=s-1, P \in H} \sum_{j \notin H} x_j \geq (2^{s-1} - 1) \cdot d.$$

The coefficient of x_P on the left hand side is 0 and for any other non-zero point $Q \neq P$ the coefficient of x_Q on the left hand side is given by

$$\frac{2^{s-1} \cdot (2^{s-1} - 1)}{2^s - 2} = 2^{s-2},$$

so that

$$n - m \geq \frac{2^{s-1} - 1}{2^{s-2}} \cdot d.$$

□

Note that the multiplicity condition m in Lemma 22 can be replaced by multiplicity condition at least m and we can also reformulate the inequality to $m \leq n - \frac{2^{s-1}-1}{2^{s-2}}$, where d can be replaced by k in the context of PIR codes.

Following up Corollary 21, we solve two further instances:

Proposition 23 *We have $P(4, 2) = 5, P(4, 4) = 9, P(4, 6) = 12, P(4, 8) = 15$, and for all even $k \geq 10$ we have $P(4, k) = P(4, k - 8\tau) + 15\tau$, where $\tau = \lfloor (k - 1)/8 \rfloor$.*

Proof For $k \in \{2, 4, 6, 8\}$ the corresponding upper bounds were known before and can be easily verified using the ILP approach. The other constructions are then obtained by combinations with a suitable number of 4-dimensional simplex codes. Except for $P(4, 4 + 8\ell) \geq 9 + 15\ell$ the coding theoretic lower bound is attained. For the latter lower bound see Corollary 16. (According to Proposition 20 it is sufficient to prove $P(4, 4) \geq 9$, which can be done using Inequality (3).) □

Proposition 24 *We have $P(5, 2) = 6, P(5, 4) = 10, P(5, 6) = 14, P(5, 8) = 18, P(5, 10) = 22, P(5, 12) = 25, P(5, 14) = 28, P(5, 16) = 31$ and for all even $k > 16$ we have $P(5, k) = P(5, k - 16\tau) + 31\tau$, where $\tau = \lfloor (k - 1)/16 \rfloor$.*

Proof For $k \in \{2, 4, 6, 16\}$ the upper bounds were known before and for $k \in \{8, 10, 12, 14\}$ the upper bounds have been found using the ILP approach, with the corresponding generator matrices:

$$\left(\begin{array}{cccccccc} 10000101010101010101 \\ 0100010011001100110011 \\ 0010001000111100001111 \\ 0001001000000011111111 \\ 00001001111111111111 \end{array} \right), \left(\begin{array}{cccccccc} 1000011101010101010010101 \\ 0100001011001111001001101 \\ 0010011000111100111000011 \\ 0001000111111100000111111 \\ 0000100000000111111111 \end{array} \right),$$

$$\left(\begin{array}{cccccccc} 1000011011001010101010101 \\ 01000101100111100110010011 \\ 001000111001111001110001111 \\ 00010000011111000001111111 \\ 000010000000001111111111 \end{array} \right), \left(\begin{array}{cccccccc} 100001101100100101 \\ 01000011010010011 \\ 001000100110001111 \\ 00010110000111111 \\ 0000100111111111 \end{array} \right).$$

The other constructions are then obtained by combinations with a suitable number of 5-dimensional simplex codes. Except for $P(5, 8+16\ell) \geq 18+31\ell, P(5, 10+16\ell) \geq 22+31\ell$, and $P(5, 12+16\ell) \geq 25+31\ell$ the coding theoretic lower bound is attained. By Proposition 20 it remains to prove $P(5, 8) \geq 18, P(5, 24) \geq 49, P(5, 10) \geq 22, P(5, 26) \geq 53$, and $P(5, 12) \geq 25$. To this end we have utilized the ILP lower bound for $\lambda = 3$, see Sect. 4. (Each computation took just a few seconds.) □

Table 1 Best known bounds for $P(s, k)$ for small parameters

s/k	2	3	4	6	8	10	12	14	16
1	c_2^e	r_3^e	r_4^e	c_6^e	c_8^e	c_{10}^e	c_{12}^e	c_{14}^e	c_{16}^e
2	c_3^e	r_5^e	r_6^e	c_9^e	c_{12}^e	c_{15}^e	c_{18}^e	c_{21}^e	c_{24}^e
3	c_4^e	r_6^e	r_7^e	c_{11}^e	c_{14}^e	c_{18}^e	c_{21}^e	c_{25}^e	c_{28}^e
4	c_5^e	r_8^I	r_9^I	c_{12}^I	c_{15}^S	c_{20}^a	i_{24}^a	c_{27}^a	c_{30}^a
5	c_6^e	r_9^I	r_{10}^I	c_{14}^I	$i_{18}^{L,I}$	i_{22}^I	i_{25}^I	c_{28}^I	c_{31}^S
6	c_7^e	r_{10}^I	r_{11}^I	c_{15}^I	i_{19}^I	c_{23}^I	i_{27}^I	i_{32}^I	i_{36}^I
7	c_8^e	r_{12}^R	r_{13}^R	c_{16}^I	c_{19-21}^I	c_{24-26}^I	c_{27-29}^I	c_{32-34}^I	c_{35-39}^I
8	c_9^e	r_{13}^R	r_{14}^R	c_{17-18}^L	c_{20-23}^L	c_{26-27}	c_{29-33}^I	c_{33-38}^I	c_{36-42}^I
9	c_{10}^e	r_{14}^R	r_{15}^R	c_{18-20}^L	c_{21-25}^R	c_{27-28}^I	c_{30-37}^I	c_{35-40}^I	c_{38-45}^I
10	c_{11}^e	r_{15}^R	r_{16}^R	c_{20-21}^L	c_{22-26}^R	c_{28-31}^L	c_{31-40}^I	c_{36-45}^I	c_{40-50}^I

We remark that also Proposition 9 gives matching constructions for $k \in \{10, 12, 14\}$.

In Table 1 we state the best known bounds for $P(s, k)$ for small parameters. Improvements over the existing literature are printed in bold.¹² We use the following letters to point to the method from which the bound was obtained.

c The coding theoretic lower bound $P(s, k) \geq N(s, k)$, see Inequality (1).

r The lower bound of Rao and Vardy, see Inequality (3).

e The exact values of $P(s, 2)$ and $P(s, k)$ for $s \leq 3$ are well-known, see Theorem 2.

a Two PIR codes can be easily combined to a new one, see Theorem 2.10)-11).

S s -dimensional simplex code.

I Constructive ILP approach, see Sect. 4. (Generator matrices can be obtained from the authors or the arXiv version of this paper.)

i ILP lower bound with $\lambda = 3$, see Sect. 4.

L Lengthening, see Proposition 7 and the subsequent algorithm, see also [18].

R Binary shortened projective Reed-Muller codes, see [28].

We remark that Table 1 contains improvements of the lower bound $P(s, k) \geq N(s, k)$ in the cases $(s, k) \in \{(4, 3), (4, 4), (4, 12), (5, 8), (5, 10), (5, 12), (6, 8), (6, 12), (6, 14)\}$. All of these can be obtained with the ILP lower bound described in Sect. 4, while for some we state a coding theoretic proof in Sect. 5.

Looking at the differences between the coding theoretic lower bound $N(s, k)$ and the best known lower bound for $P(s, k)$ it seems that $P(s, 2^{s-2})$ is an instance where the optimal codes have a large length compared to the coding theoretic lower bound.

Conjecture 25

$$\lim_{s \rightarrow \infty} P(s, k) - N(s, k) = \infty,$$

where $k = 2^{s-2}$.

Prescribing the cyclic group \mathbb{Z}_s as a subgroup of the automorphism group we have obtained the following upper bounds for these instances, where the coding theoretic lower bound is attained by the first order Reed-Muller codes:

¹² Note that this can be slightly ambiguous, if the exact value was determined. In [18] the upper bound $P(5, 8) \leq 18$ was obtained and we only improved the lower bound.

- $P(6, 16) \leq 36$; cardinalities of the recovery sets: $1^1 2^{10} 3^5$
- $P(7, 32) \leq 71$; cardinalities of the recovery sets: $1^1 2^{23} 3^8$
- $P(8, 64) \leq 142$; cardinalities of the recovery sets: $1^1 2^{48} 3^{15}$
- $P(9, 128) \leq 282$; cardinalities of the recovery sets: $1^1 2^{100} 3^{27}$

As a general construction we may use Proposition 9.

Corollary 26 For $s \geq 5$ we have

$$P(s, 2^{s-2}) \leq 2^s - 1 - 3 \cdot 2^{s-3} = 5 \cdot 2^{s-3} - 1.$$

Note that this gives $P(5, 8) \leq 19$, $P(6, 16) \leq 39$ (which is also an improvement to [11, Table I]), $P(7, 32) \leq 79$, $P(8, 64) \leq 159$, and $P(9, 128) \leq 319$. In general we have $P(s, 2^{s-2}) \geq N(s, 2^{s-2}) = 2^{s-1} = 4 \cdot 2^{s-3}$.

Appendix A: Generator Matrices for the Stated Best Known Upper Bounds on $P(s, k)$

In this section we state generator matrix for the currently best known codes found by the ILP approach from Sect. 4, where we only consider the cases of dimension $s \geq 2$ and demand $k \geq 4$. Note that all codes are projective and except for the cases of $P(6, 8)$, $P(6, 10)$, $P(6, 12)$, $P(7, 12)$, $P(9, 16)$ all generator matrices are systematic.

$P(4, 4) \leq 9$

$$\begin{pmatrix} 100011111 \\ 010001011 \\ 001011001 \\ 000100111 \end{pmatrix}$$

$P(4, 6) \leq 12$

$$\begin{pmatrix} 100011010101 \\ 010010110011 \\ 001010001111 \\ 000101111111 \end{pmatrix}$$

$P(5, 4) \leq 10$

$$\begin{pmatrix} 1000011111 \\ 0100010101 \\ 0010011001 \\ 0001001011 \\ 0000100111 \end{pmatrix}$$

$P(5, 6) \leq 14$

$$\begin{pmatrix} 10000101110001 \\ 01000110010011 \\ 00100010101011 \\ 00010001100111 \\ 00001000011111 \end{pmatrix}$$

$$P(5, 8) \leq 18$$

$$\begin{pmatrix} 100001101100100101 \\ 010000111010010011 \\ 001000100110001111 \\ 000101100001111111 \\ 000010011111111111 \end{pmatrix}$$

$$P(5, 10) \leq 22$$

$$\begin{pmatrix} 10000101010101010101 \\ 0100010011001100110011 \\ 0010001000111100001111 \\ 0001001000000011111111 \\ 0000100111111111111111 \end{pmatrix}$$

$$P(5, 12) \leq 25$$

$$\begin{pmatrix} 1000011101010101010010101 \\ 0100001011001111001001101 \\ 0010011000111100111000011 \\ 0001000111111100000111111 \\ 000010000000011111111111 \end{pmatrix}$$

$$P(5, 14) \leq 28$$

$$\begin{pmatrix} 1000011011001010101010101 \\ 0100010110100111100110010011 \\ 0010001110011110011110001111 \\ 0001000001111110000001111111 \\ 00001000000000111111111111 \end{pmatrix}$$

$$P(6, 4) \leq 11$$

$$\begin{pmatrix} 10000010011 \\ 01000011100 \\ 00100011001 \\ 00010001101 \\ 00001001011 \\ 00000100111 \end{pmatrix}$$

$$P(6, 6) \leq 15$$

$$\begin{pmatrix} 100000011111011 \\ 010000110011001 \\ 001000010100111 \\ 000100001101110 \\ 000010111100001 \\ 000001000011111 \end{pmatrix}$$

$$P(6, 8) \leq 19$$

$$\begin{pmatrix} 0101111011110111011 \\ 1111010111101101110 \\ 1111001111011111001 \\ 1100111110111100111 \\ 1100000001111111111 \\ 0011111111111100000 \end{pmatrix}$$

$$P(6, 10) \leq 23$$

$$\begin{pmatrix} 110110101101111011011 \\ 011101111011010110110 \\ 110011111110011100011 \\ 111111001110000011111 \\ 001111100000111111111 \\ 000000011111111111111 \end{pmatrix}$$

$$P(6, 12) \leq 27$$

$$\begin{pmatrix} 101101101101101101101 \\ 011011011011011011011 \\ 111111111000111000111 \\ 111000111111110000001 \\ 000111111000000111111 \\ 000000000111111111111 \end{pmatrix}$$

$$P(6, 14) \leq 32$$

$$\begin{pmatrix} 10000010010101101010001 \\ 01000001001100100110100 \\ 00100010100011100001100 \\ 00010001100000011111100 \\ 00001001100000000000011 \\ 00000100011111111111111 \end{pmatrix}$$

$$P(6, 16) \leq 36$$

$$\begin{pmatrix} 10000010100011010110101 \\ 01000001110110101101111 \\ 00100011101110011111100 \\ 00010000011110000011000 \\ 00001000000011111110000 \\ 00000100000000000000011 \end{pmatrix}$$

$$P(7, 6) \leq 16$$

$$\begin{pmatrix} 1000000110100110 \\ 0100000010101011 \\ 0010000011110001 \\ 0001000110010011 \\ 0000100111101000 \\ 0000010001100111 \\ 0000001000011111 \end{pmatrix}$$

$$P(7, 8) \leq 21$$

$$\begin{pmatrix} 10000010100111010111 \\ 01000001001111101101 \\ 00100000111010111011 \\ 000100010101011100001 \\ 000010001100110011001 \\ 0000010000111110000111 \\ 0000001000000011111111 \end{pmatrix}$$

$$P(7, 10) \leq 26$$

$$\begin{pmatrix} 10000001010000101011010101 \\ 01000000100010110101110001 \\ 00100000001100101100101101 \\ 00010001010101111111100011 \\ 00001000110011111100011111 \\ 00000100001111100011111111 \\ 00000010000000011111111111 \end{pmatrix}$$

$$P(7, 12) \leq 29$$

$$\begin{pmatrix} 1111101101111101111011010101 \\ 11110110111110111110110110011 \\ 11011111110101111011110001111 \\ 10111111110011110110001111111 \\ 01111110001111110001111111111 \\ 0000111000000001111111111111 \\ 0000000111111111111111111111 \end{pmatrix}$$

$$P(7, 14) \leq 34$$

$$\begin{pmatrix} 1000000110100011010000100011111001 \\ 0100000101110010000101101110000011 \\ 0010000011101000010100011000101111 \\ 0001000000010110110011100110011001 \\ 0000100000001110001110011110000111 \\ 0000010000000011111100000011111111 \\ 0000001000000000000011111111111111 \end{pmatrix}$$

$$P(7, 16) \leq 39$$

$$\begin{pmatrix} 100000011011010000001011111000001011010 \\ 010000010000110101001000101011111000101 \\ 001000010110000011001110000110100110011 \\ 000100001110001000100110011110011110000 \\ 000010000001111000010001111110000001111 \\ 000001000000000111110000000001111111111 \\ 00000010000000000000111111111111111111 \end{pmatrix}$$

$P(7, 32) \leq 71$

$$\begin{pmatrix} 10000001011010101101010101001000100101011100011010111101010100101000001 \\ 01000001110001011100110011011101101100110101001001010100000010000101011 \\ 00100000110111000100001100111100011100000011000111000011001110011100111 \\ 0001000000111100001111110000100000011110000111111001111000001111100001 \\ 00001000000000111111111100000011111111110000000000111111000000000011111 \\ 00000100000000000000000011111111111111110000000000000000111111111111111 \\ 000000100000000000000000000000000000000011111111111111111111111111111111 \end{pmatrix}$$

$P(8, 6) \leq 18$

$$\begin{pmatrix} 100000011010011000 \\ 01000001010101100 \\ 001000001111000100 \\ 000100011001001100 \\ 00001001111010000 \\ 000001000110011100 \\ 000000100001111100 \\ 001000010011000011 \end{pmatrix}$$

$P(8, 8) \leq 23$

$$\begin{pmatrix} 10000001010011101011100 \\ 01000000100111110110100 \\ 00100000011101011101100 \\ 00010001010101110000100 \\ 00001000110011001100100 \\ 00000100001111000011100 \\ 00000010000000111111100 \\ 0000000111111111110011 \end{pmatrix}$$

$P(8, 10) \leq 27$

$$\begin{pmatrix} 100000001010100010110100101 \\ 010000001110010110000101001 \\ 001000000101010011010011001 \\ 000100000011001001110001011 \\ 000010001010011000011000111 \\ 000001000110110001001100101 \\ 000000100001110111000100011 \\ 000000010000001111000011111 \end{pmatrix}$$

$P(8, 12) \leq 33$

$$\begin{pmatrix} 100000001110110011010001010011001 \\ 010000000001101110001101011100011 \\ 001000001100100011111010101001001 \\ 000100001011011001001000011100111 \\ 000010001001000111110110000111001 \\ 000001000111000000110001111100111 \\ 000000100000111111110000000111111 \\ 000000010000000000001111111111111 \end{pmatrix}$$

$$P(8, 14) \leq 38$$

$$\begin{pmatrix} 10000000110111001001010110100100000100 \\ 01000000011001100110100000110001010011 \\ 00100000010100110001101100001110100100 \\ 00010000000011101100011001100001101010 \\ 00001000110000000011111010011100011000 \\ 000001000011111100000000110000011111001 \\ 00000010000000011111111111000000000111 \\ 00000001000000000000000001111111111111 \end{pmatrix}$$

$$P(8, 16) \leq 42$$

$$\begin{pmatrix} 10000000110010110000010010101100011100110 \\ 01000000101110100101100000101101000101100 \\ 001000001010111011010011001010100001100000 \\ 000100000110011001110101001001101100000001 \\ 000010000001111000110011111000010100000011 \\ 000001000000000111110000111000001100011111 \\ 00000010000000000001111111000000011111111 \\ 0000000100000000000000000111111111111111 \end{pmatrix}$$

$$P(9, 10) \leq 28$$

$$\begin{pmatrix} 1000000001010100010110100101 \\ 0100000001110010110000101001 \\ 0010000000101010011010011001 \\ 0001000000011001001110001011 \\ 0000100001010011000011000111 \\ 0000010000110110001001100101 \\ 0000001000001110111000100011 \\ 0000000100000001111000011111 \\ 0000000010000000000111111111 \end{pmatrix}$$

$$P(9, 12) \leq 37$$

$$\begin{pmatrix} 1000000001111001001000001100100101001 \\ 0100000001001110110001001010100000011 \\ 0010000000101100010110110000001001011 \\ 0001000000000010011100010001110011101 \\ 0000100000011001000010010111100100101 \\ 0000010001111000110001001000010100101 \\ 0000001000000111110000111000001100011 \\ 000000010000000000111111100000011111 \\ 000000001000000000000000011111111111 \end{pmatrix}$$

$$P(9, 14) \leq 40$$

```
(
100000001100010011000110101100110000001
010000001111000001100101010000000110101
0010000000011101000000000111010100101011
0001000001000111101010000100100010000111
0000100000110000100111100010001110000101
0000010000001100011001100000011101100011
00000010000000111111000011110000011100001
0000000100000000000111111110000000111111
00000000100000000000000000011111111111111)

```

$$P(9, 16) \leq 45$$

```
(
1010100010010010000110000100011010101000010101
111001011000011001001010000010010010000101000
010111000100111000000110001000110011000000001
001110110100001100110110000000000101000010010
000001110011100100010011000110001101000001000
000000001111100011110001000010000011010000101
00000000000001111111000011111000000110000011
00000000000000000000111111110000000011111111
00000000000000000000000000000111111111111111111)

```

$$P(10, 10) \leq 31$$

```
(
1000000001010100010110100101000
0100000001110010110000101001000
0010000000101010011010011001000
0001000000011001001110001011000
0000100001010011000011000111000
0000010000110110001001100101000
0000001000001110111000100011000
0000000100000001111000011111000
000000001000000000011111111000
00000000010001111011000000111)

```

$$P(10, 12) \leq 40$$

```
(
1000000000110001100110101000100010000010
0100000000001100100001010001010110100001
0010000000100011110010010000001001000110
0001000000011000001101010100110011000000
0000100000100110100100000011001001010001
0000010000011000011100110001000000001110
0000001000000110000000001111100111000100
0000000100000001111100000000100000111101
000000001000000000001111111110000000011
0000000001000000000000000000111111111111)

```

$$P(10, 14) \leq 45$$

$$\left(\begin{array}{l} 100000000010100100100110100101000110101001000 \\ 010000000001001100110010010100110011000010010 \\ 001000000010010001110100011000010100011010001 \\ 000100000011000110000001111100100100010101000 \\ 000010000001010100011000100001001011010110001 \\ 000001000000110010110100101011000101000001010 \\ 000000100000001110001100011000100011001100110 \\ 000000010000000001111100000111100000111100001 \\ 00000000100000000000001111111110000000011111 \\ 00000000010000000000000000000000000111111111111 \end{array} \right)$$

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