Coding for the ℓ_∞-Limited Permutation Channel

Michael Langberg
Electrical Engineering
SUNY at Buffalo
Buffalo, NY, USA
mikel@buffalo.edu

Moshe Schwartz
Electrical and Computer Engineering
Ben-Gurion University of the Negev
Beer Sheva 8410501, Israel
schwartz@ee.bgu.ac.il

Eitan Yaakobi
Computer Science
Technion – Israel Institute of Technology
Haifa 32000, Israel
yaakobi@cs.technion.ac.il

***Abstract*—In this work we consider the communication of information in the presence of synchronization errors. Specifically, we consider *permutation channels* in which a transmitted codeword $x = (x_1, \ldots, x_n)$ is corrupted by a permutation $\pi \in S_n$ to yield the received word $y = (y_1, \ldots, y_n)$ where $y_i = x_{\pi(i)}$. We initiate the study of worst case (or zero error) communication over permutation channels that distort the information by applying permutations π which are limited to displacing any symbol by at most r locations, i.e. permutations π with weight at most r in the ℓ_∞-metric. We present direct and recursive constructions, as well as bounds on the rate of such channels for binary and general alphabets. Specific attention is given to the case of $r = 1$.**

I. Introduction

Permutation channels have received some attention in recent years due to their relevance in different applications of networking technologies and various read channels. Under this setup, a vector of symbols is transmitted in some order, but due to synchronization errors, the symbols received are not necessarily in the order in which they were transmitted, e.g., [7], [8], [20] (permutation channels), [9], [17] (the bit-shift magnetic recording channel), and [13] (the Trapdoor channel).

We can think of the channel as applying a permutation to the transmitted vector. However, not all permutations may be equally likely, or even feasible. In this work we focus on channels that can only displace symbols a limited amount of positions away from their origin. Such permutations are exactly those that have a limited weight in the ℓ_∞-metric over permutations.

When the transmitted vectors are in themselves permutations, this channel has been studied as the limited-magnitude rank-modulation channel. In particular, error-correcting codes were studied [6], [18], [19], as well as systematic codes [22], anticodes [16], covering codes [3], [21], and various other related combinatorial problems [5], [11], [15].

Unlike the rank modulation case, this work considers the transmission of general vectors over the channel, and in particular, allows repeated symbols and small alphabets. More specifically, for a finite alphabet Σ, the transmitted codeword $x = (x_1, \ldots, x_n)$ may be any element in Σ^n. The codeword x is corrupted by a permutation $\pi \in S_n$ to yield $y = (y_1, \ldots, y_n)$ where $y_i = x_{\pi(i)}$. We consider the worst case (or zero error) communication model over permutations π for which $\forall i: |i - \pi(i)| \leqslant r$ for a pre-specified magnitude r, i.e., π is r-bounded by the ℓ_∞-metric. We refer to such channels as ℓ_∞-limited permutation channels, $\mathrm{LPC}_\infty(r)$.

In this work we initiate the study of $\mathrm{LPC}_\infty(r)$ for general alphabets Σ and magnitudes r under the zero error (worst case) setting. Although similar models have been studied in the literature, to the best of our knowledge, the study of zero error $\mathrm{LPC}_\infty(r)$ has not been explicitly addressed. Most closely related models include the permutation model of [9], [17] in which $\Sigma = \{0, 1\}$ but the limitation $|i - \pi(i)| \leqslant r$ on permutations π holds only for i such that $x_i = 1$, [8] which has a model similar to ours but applies a random permutation instead of a worst case one, [7] in which random synchronization errors of limited ℓ_∞-norm are applied to vectors of natural numbers, and [20] in which the channel is governed by a distribution over S_n.

We present direct and recursive code constructions, encoding and decoding algorithms, bounds on code parameters, and constructions for covering codes for $\mathrm{LPC}_\infty(r)$. Specifically, our model and preliminaries are given in Section II. In Section III we study the combinatorial properties of $\mathrm{LPC}_\infty(r)$ including the average and precise size of balls according to the ℓ_∞-metric. In Section IV we present codes for $\mathrm{LPC}_\infty(r)$. Finally, in Section V we present general upper bounds on the size of codes for $\mathrm{LPC}_\infty(r)$ via covering codes together with the comparison of our lower and upper bounds for some specific settings of parameters. Our main focus in several of the sections above is on general $|\Sigma| = q$ and $r = 1$, and only at times do we address larger values of r. Due to the lack of space some of the proofs in the paper are omitted.

II. Preliminaries

Let us denote $[n] = \{1, 2, \ldots, n\}$, and let S_n denote the set of all permutations over $[n]$. A permutation $\pi \in S_n$ is written in vector notation $\pi = [\pi_1, \pi_2, \ldots, \pi_n]$, and may be considered a bijection $\pi : [n] \to [n]$ mapping $\pi(i) = \pi_i$. The identity permutation is denoted by $\mathrm{Id} = [1, 2, \ldots, n]$.

Given two permutations, $\pi, \pi' \in S_n$, the ℓ_∞-distance between the two is defined as

$$d_\infty(\pi, \pi') = \max_{i \in [n]} \left|\pi(i) - \pi'(i)\right|.$$

The ℓ_∞-distance defines a metric [2]. The *weight* of a permutation $\pi \in S_n$ is defined as

$$\mathrm{wt}(\pi) = d_\infty(\pi, \mathrm{Id}).$$

Authors appear in alphabetical order. The first author was supported in part by BSF grant 2010075, the second author by the Israel Science Foundation (ISF) grant No. 130/14, and the third author by the Israel Science Foundation (ISF) grant No. 1624/14.

Thus, all the permutations of weight at most r form the ball of radius r centered at the identity permutation. Balls in the ℓ_∞-metric over permutations have been studied in the past [5], [11], [15].

We now formally introduce the ℓ_∞-limited permutation channel, $\text{LPC}_\infty(r)$.

Definition 1. *Let Σ be some finite alphabet. Assume a vector $x = (x_1, x_2, \dots, x_n) \in \Sigma^n$ has been transmitted. The $\text{LPC}_\infty(r)$ channel distorts it by applying to it a permutation of weight at most r. Thus, the received vector $y = (y_1, y_2, \dots, y_n) \in \Sigma^n$ satisfies $y = \pi x$, i.e.,*

$$y_i = x_{\pi(i)} \qquad \text{for all } i \in [n],$$

for some permutation $\pi \in S_n$ with $\text{wt}(\pi) \leqslant r$.

Definition 2. *The* ball *of radius r centered at $x \in \Sigma^n$ is*

$$B_r(x) = \{ y \in \Sigma^n \mid y = \pi x, \pi \in S_n, \text{wt}(\pi) \leqslant r \}.$$

It follows that a vector $x \in \Sigma^n$ transmitted over $\text{LPC}_\infty(r)$ may be received as any vector in $B_r(x)$. This gives rise to the following definition of an error-correcting code for $\text{LPC}_\infty(r)$.

Definition 3. *Let Σ be a finite alphabet of size q, and $C \subseteq \Sigma^n$. We say C is an $(n, M; r)_q$-LPC_∞ code if its size is $|C| = M$, and for all $c, c' \in C$, $c \neq c'$, we have*

$$B_r(c) \cap B_r(c') = \varnothing.$$

In an analogous fashion we also define covering codes.

Definition 4. *Let Σ be a finite alphabet and $C \subseteq \Sigma^n$. We say C is an $(n, M)_q R$-LPC_∞ covering code if its size is $|C| = M$, and*

$$\bigcup_{c \in C} B_R(c) = \Sigma^n.$$

The sizes of the largest code, and the smallest covering code, are now defined. We use a notation similar to [1], [12].

Definition 5. *Let $\Sigma = \mathbb{Z}_q$ be the alphabet. Given n and r, we denote by $A_q(n; r)$ the largest M such that there exists an $(n, M; r)_q$-LPC_∞ code over Σ. Similarly, given n and R, we denote by $K_q(n; R)$ the smallest M such that there exists an $(n, M)_q R$-LPC_∞ covering code over Σ.*

Let Σ be some finite alphabet. We recall some useful notation commonly used in the theory of formal languages. An n-string $x = x_1 x_2 \cdots x_n \in \Sigma^n$ is a finite sequence of alphabet symbols, $x_i \in \Sigma$. We say n is the length of x and denote it by $|x| = n$. For two strings, $x \in \Sigma^n$ and $y \in \Sigma^m$, their concatenation is denoted by $xy \in \Sigma^{n+m}$. The set of all finite strings over the alphabet Σ is denoted by Σ^*. For $s \in \Sigma^*$ and a non-negative integer k, we use s^k to denote the sequence obtained by concatenating k copies of s.

III. Properties of the $\text{LPC}_\infty(r)$ Space

In this section we study several properties of the $\text{LPC}_\infty(r)$ space, including the size of balls, and the distance between vectors.

Definition 6. *The* LPC_∞-distance *between $x, y \in \Sigma^n$, denoted $d(x, y)$, is defined as the minimum non-negative integer w such that there exists a permutation $\pi \in S_n$, $\text{wt}(\pi) = w$, and $y = \pi x$. If no such integer exists we say the distance is ∞.*

We note that the distance function is symmetric since $\text{wt}(\pi) = \text{wt}(\pi^{-1})$ for all $\pi \in S_n$. Additionally, $d(x, y) = 0$ if and only if $x = y$. It is obvious that $d(x, y) \neq \infty$ if and only if every symbol of the alphabet appears in x the exact same times as it appears in y. Finally, the triangle inequality holds, $d(x, y) \leqslant d(x, z) + d(z, y)$, for all x, y, and z, provided none of the distances in the inequality is ∞. To sketch the proof for this, assume the minimal-weight permutations that determine the distances are $\pi_1 x = z$, $\pi_2 z = y$, and $\pi x = y$. Then,

$$\begin{aligned} d(x,z) + d(z,y) &= \text{wt}(\pi_1) + \text{wt}(\pi_2) = \text{wt}(\pi_1) + \text{wt}(\pi_2^{-1}) \\ &\geqslant \text{wt}(\pi_1 \pi_2) \geqslant \text{wt}(\pi) = d(x,y), \end{aligned}$$

where the first inequality is due to the triangle inequality in the ℓ_∞-metric over permutations, and the second is due to the fact that $\pi_1 \pi_2 x = y$, but $\pi_1 \pi_2$ is not necessarily the minimal-weight permutation changing x into y. Thus, when restricting ourselves to sets of vectors with the same composition, the distance function d defines a metric.

To calculate the LPC_∞-distance we use the following notation. For any symbol $a \in \Sigma$, we denote by $n_a(x)$ the number of occurrences of a in x, i.e.,

$$n_a(x) = |\{ i \in [n] \mid x_i = a \}|.$$

Additionally, the index of the jth occurrence of a in x is denoted as $L_a(j; x)$. More precisely, $L_a(j; x) = i$ if $x_i = a$ and a appears $j - 1$ times in the string $x_1 x_2 \dots x_{i-1}$. Using this notation, our previous observation becomes: $d(x, y) < \infty$ iff $n_a(x) = n_a(y)$ for all $a \in \Sigma$.

The next theorem states how to find $d(x, y)$ and the corresponding permutation connecting x and y.

Theorem 7. *Let Σ be a finite alphabet, and $x, y \in \Sigma^n$ such that $d(x, y) < \infty$. Then*

$$d(x, y) = \max_{\substack{a \in \Sigma \\ j \in [n_a(x)]}} |L_a(j; x) - L_a(j; y)|.$$

In addition, finding π such that $y = \pi x$ and $\text{wt}(\pi) = d(x, y)$ can be done in $O(n)$ time.

On several occasions in the following sections, we will focus specifically on $(n; 1)_q$-LPC_∞ codes. We therefore study in more detail balls of radius 1. Assume the alphabet is $\Sigma = \mathbb{Z}_q$, and let $x = (x_1, x_2, \dots, x_n) \in \mathbb{Z}_q^n$ be some vector. The number of permutations $\pi \in S_n$ such that $\text{wt}(\pi) \leqslant 1$ is known to be the n-th Fibonacci number F_n (see [11], [15]), where

$$F_i = \begin{cases} F_i = F_{i-1} + F_{i-2} & i \geqslant 2 \\ F_i = 1 & i = 0, 1 \end{cases}$$

Thus, we immediately get that

$$|B_1(x)| \leqslant F_n.$$

However, it is also clear that applying distinct permutations to x does not always result in distinct vectors.

Given two permutations $\pi, \pi' \in S_n$, $\text{wt}(\pi) = \text{wt}(\pi') \leqslant 1$, we say they are x-equivalent, denoted $\pi \overset{x}{\sim} \pi'$, if $\pi x = \pi' x$. It is obvious that $\overset{x}{\sim}$ is an equivalence relation, and that $|B_1(x)|$ is the number of equivalence classes of $\overset{x}{\sim}$.

We also note that every permutation $\pi \in S_n$ with $\mathrm{wt}(\pi) \leqslant 1$ can be written uniquely as a product of non-overlapping adjacent transpositions, and more precisely,

$$\pi = \prod_{i\in[k]} (j_i, j_i + 1), \tag{1}$$

with $j_i + 1 < j_{i+1}$ for all $i \in [k-1]$. Here, (a, b), $a \neq b$, denotes in cycle notation the permutation exchanging a and b while fixing all other elements. Additionally, Π, as it appears in (1), when applied to permutations, denotes permutation composition.

We also introduce a new operator on permutations, turning each permutation π as in (1) into an *x-reduced form*,

$$\mathrm{rdc}_x(\pi) = \prod_{\substack{i\in[k] \\ x_{j_i}\neq x_{j_i+1}}} (j_i, j_i + 1).$$

Intuitively, the x-reduced form of π keeps only those transpositions that switch the positions of distinct symbols in x. By definition we have the following simple observation: for every $x \in \mathbb{Z}_q^n$ and every $\pi \in S_n$, $\mathrm{wt}(\pi) \leqslant 1$, we have

$$\pi x = \mathrm{rdc}_x(\pi)x. \tag{2}$$

In addition, the operator $\mathrm{rdc}_x(\cdot)$ characterizes the equivalence relation $\overset{x}{\sim}$.

Lemma 8. *For $x \in \mathbb{Z}_q^n$, $\pi, \pi' \in S_n$, $\mathrm{wt}(\pi), \mathrm{wt}(\pi') \leqslant 1$, we have $\pi \overset{x}{\sim} \pi'$ if and only if $\mathrm{rdc}_x(\pi) = \mathrm{rdc}_x(\pi')$.*

It now follows that $|B_1(x)|$ is exactly the number of x-reduced permutations. As already observed, x-reduced permutations are uniquely defined by a product of non-overlapping adjacent transpositions, exchanging positions in x with distinct symbols.

Given a vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{Z}_q^n$, an *antirun* of length $\ell + 1$ is a subsequence $(x_j, x_{j+1}, \ldots, x_{j+\ell})$ such that $x_{j+i} \neq x_{j+i-1}$ for all $i \in [\ell]$. A *maximum antirun* is an antirun that cannot be extended in either direction. Any sequence of $x \in \mathbb{Z}_q^n$ can be partitioned uniquely into a sequence of maximal antiruns. We call the sequence of the lengths of the maximal antiruns in such a partition, the *antirun profile of x*, and denote it as $\mathcal{P}(x)$.

Example 9. *Let $\Sigma = \mathbb{Z}_3$, and take*

$$x = (1, 1, 2, 0, 1, 0, 2, 2, 2, 2, 0, 0, 1, 2, 0).$$

We note that $(x_3, x_4, x_5) = (2, 0, 1)$ is an antirun, however it is not a maximal antirun since it may be extended. The partition of x into maximal antiruns produces

$$(1), (1, 2, 0, 1, 0, 2), (2), (2), (2, 0), (0, 1, 2, 0).$$

Thus, the antirun profile of x is

$$\mathcal{P}(x) = (1, 6, 1, 1, 2, 4).$$

□

Theorem 10. *Let $x \in \mathbb{Z}_q^n$ be a vector with an antirun profile $\mathcal{P}(x) = (\ell_1, \ell_2, \ldots, \ell_k)$. Then $|B_1(x)| = \prod_{i\in[k]} F_{\ell_i}$.*

We are interested in finding the extreme cases of the size of radius-1 balls. Since for every $x \in \mathbb{Z}_q^n$, the sum of the entries in $\mathcal{P}(x)$ is also n, to find the maximum size of $|B_1(x)|$, we are interested in finding an integer partition of n, say $(\ell_1, \ell_2, \ldots, \ell_k)$, $\ell_i \geqslant 1$, $\sum_{i\in[k]} \ell_k = n$, such that $\prod_{i\in[k]} F_{\ell_i}$ is maximized. The following identity on Fibonacci numbers is well known,

$$F_{a+b} = F_a F_{b+1} + F_{a-1} F_b.$$

A simple rearrangement of this equality, also using the basic recursion, we get

$$F_{a+b} = 2F_a F_b + F_a F_{b-1} - F_{a-2} F_b > F_a F_b,$$

for all $a, b \geqslant 1$. Thus,

Theorem 11. *The maximum radius-1 ball size is obtained when $x \in \mathbb{Z}_q^n$ is made of a single maximal antirun, and then $|B_1(x)| = F_n$. There are exactly $q(q-1)^{n-1}$ such vectors x. Conversely, the minimum size of a ball is easily seen to be obtained when $x \in \mathbb{Z}_q^n$ is comprised of a single repeating symbol from $\mathbb{Z}_q$, and then $|B_1(x)| = 1$, and there are q such vectors x.*

Extensions for the *average* size of balls are also of interest in code design and bounds, and will be included in the long version of this paper.

IV. Code Constructions

In this section we present two constructions of different flavor. The first is a direct construction, inspired by constrained-coding theory. In contrast, the second construction is recursive and requires seed codes.

A. Direct construction

The direct construction we present focuses on the binary case. As we shall later see, the rate of any binary $(n, M; 1)$-LPC_∞ is *asymptotically* at most $2/3$. Thus, we are interested in finding codes with rate as close as possible to this upper bound. Our approach to constructing such codes is motivated by a similar problem which was studied by Shamai and Zehavi in [17] and later by Krachkovsky in [9]. The problem studied by them is an asymmetric version of the binary channel studied here. While in the binary model of the $\mathrm{LPC}_\infty(r)$ channel, every bit can change its location by at most r positions, in the model studied in [9], [17] this constraint is applied only to the bits having value 1. For example, for the word $x = 000111$, the ball of radius 1 under the $\mathrm{LPC}_\infty(1)$ channel is $\{000111, 001011\}$, whereas in the asymmetric version of this channel, it is the set $\{000111, 001011, 001101, 001110\}$.

Let us fix from now on $\Sigma = \{0, 1\}$. The construction in [9], [17] consists of the following idea. Given a set of blocks $\mathcal{B} \subseteq \Sigma^*$ (these blocks can be of any length), the code $C_n(\mathcal{B})$ is defined to be

$$C_n(\mathcal{B}) = \left\{ b_1 \ldots b_m \;\middle|\; b_1, \ldots, b_m \in \mathcal{B}, \sum_{i=1}^{m} |b_i| = n \right\}. \tag{3}$$

Under this construction it is possible to derive that the asymptotic rate of this code family will be given by $\log_2 \lambda$, i.e.,

$$\limsup_{n\to\infty} \frac{\log_2 |C_n(\mathcal{B})|}{n} = \log_2 \lambda,$$

where λ is the largest solution of the equation

$$\sum_{b\in\mathcal{B}} x^{-|b|} = 1.$$

The main goal of the works [9], [17] was to study the asymmetric version of the $\mathrm{LPC}_\infty(r)$ channel but for codes that satisfy the *run-length limited (RLL)* constraint [4]. However, as a special case, one can have no RLL constraint. In this case the set of blocks $\mathcal{B} = \left\{0^{3i}1 \mid i \geqslant 0\right\}$ was used to generate a code family with asymptotic rate ≈ 0.551 which attains the capacity of the constraint. Since the error balls in the $\mathrm{LPC}_\infty(r)$ distortion channel are subsets of the error balls of the asymmetric version of this channel model, we could also take the same set $\mathcal{B}$ and thus achieve at least the same rate. Next, we will show how to improve upon this construction and get an asymptotic rate of ≈ 0.5875.

Construction A. *Define the block set* $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4$, *where*

$$\mathcal{B}_1 = \left\{0^{2+3i}1 \mid i \geqslant 0\right\}, \quad \mathcal{B}_2 = \left\{0^{3+3i}1^4 \mid i \geqslant 0\right\},$$
$$\mathcal{B}_3 = \left\{1^{2+3i}0 \mid i \geqslant 0\right\}, \quad \mathcal{B}_4 = \left\{1^{3+3i}0^4 \mid i \geqslant 0\right\}.$$

The constructed code is $C_n(\mathcal{B})$ *as defined in* (3). □

Theorem 12. *For all* $n \geqslant 3$, *the code* $C_n(\mathcal{B})$ *from Construction* A *is an* $(n;1)$-LPC_∞ *code, and allows decoding in time* $\Theta(n)$.

Corollary 13. *The asymptotic rate of the code family* $C_n(\mathcal{B})$ *from Construction* A *is* $\log_2 \lambda \approx 0.5875$, *where* λ *is the largest solution of the equation* $x^7 - 3x^4 - 2 = 0$.

The proof of Corollary 13 follows standard techniques (see, e.g., [14]). We also mention that in the other extreme case in which $q = |\Sigma|$ is large (w.r.t. r) we have codes with rate approaching 1. Such codes are reminiscent of network protocols that add meta-data to packets in order to correct packets that arrive out of order.

Theorem 14. *For* $(2r+1)|q$, *there exist* $(n, M; r)_q$-LPC_∞ *codes with* $M = \left(\frac{q}{2r+1}\right)^n$ *(and thus rate* $1 - \log_q(2r+1)$*).*

B. Recursive construction

We present a recursive construction that may be combined with seed codes either from a direct construction or from a computer search.

Construction B. *Let* C *be an* $(n, M; 1)_q$-LPC_∞ *code over the alphabet* $\Sigma = \mathbb{Z}_q$ *for some integer* $q \geqslant 2$. *Define*

$$C_a = \left\{ c = (c_1, c_2, \dots, c_n) \in C \;\middle|\; \sum_{i=1}^{n} c_i \equiv a \pmod q \right\}.$$

Obviously $C_0, C_1, \dots, C_{q-1}$ *form a partition of* C.

Let $a_1, a_2, \dots, a_\ell \in \mathbb{Z}_q$ *be a sequence of integers such that* $a_i \neq a_{i+1}$, *for all* $1 \leqslant i \leqslant \ell - 1$, *or in the notation of the previous section, the sequence is a single antirun. The constructed code is*

$$C' = C_{a_1} \times C_{a_2} \times \cdots \times C_{a_\ell}.$$

We say all the words of C_{a_i} *have a sum of* a_i, *and we say* $(a_1, a_2, \dots, a_\ell)$ *is the* signature *of all the words in* C'. □

Theorem 15. *Let* C *be as in Construction* B. *Then the code* C' *from Construction* B *is an* $(n\ell, M'; 1)_q$-LPC_∞ *code, with* $M' = \prod_{i=1}^{\ell} |C_{a_i}|$.

Example 16. *Consider the binary* $(3,4;1)_2$-LPC_∞ *code* $C = \{000, 100, 110, 111\}$. *By Section* V *this code is optimal. Using Construction* B *we can create a family of* LPC_∞ *codes with parameters* $(6n, 4^m; 1)_2$ *for all* $m \geqslant 1$. *All the codes in this family have rate* $\frac{1}{3}$. □

Example 17. *We can construct codes using a greedy computer search in the following manner. Fix an alphabet, in this case,* $\Sigma = \mathbb{Z}_2$. *Set a length* n, *and write a lexicographic list of all the length-*n *vectors over* Σ. *Start with an empty set* C^0. *At the* i*th iteration,* $i = 1, 2, \dots$, *find the lexicographically-least vector* $c \in \Sigma^n \setminus C^{i-1}$ *such that* $C^{i-1} \cup \{c, \overline{c}\}$ *is still an* $(n;1)_2$-LPC_∞ *code, where* $\overline{c}$ *denotes the bit-wise complement of* c.

Using such a procedure, for length $n = 24$ *a computer search resulted in an* LPC_∞*-code* C *with parameters* $(24, 50220; 1)_2$. *This code has rate* ≈ 0.650667. *The code* C *has 25122 codewords of even weight, and 25098 codewords of odd weight. Using Construction* B *we can create a family of* LPC_∞*-codes with parameters* $(48m, 630511956^m; 1)_2$ *for all* $m \geqslant 1$. *All the codes in the family have rate* ≈ 0.608999 *(which are the highest rate binary codes presented in this work).* □

V. Bounds on Code Parameters

We now present a number of upper bounds on code size. This first theorem shows a connection between $A_q(n;r)$ and $K_q(n;r)$. While this connection is well-known in other settings, the usual techniques of proving it do not work here since the size of balls depends on their center. Nevertheless, the proof is elementary.

Theorem 18. *For all* n *and* r,

$$A_q(n;r) \leqslant K_q(n;r).$$

This simple argument implies the following general bound.

Theorem 19. *Let* $\Sigma = \mathbb{Z}_q$ *be the alphabet,* $q \geqslant 2$. *Then for all* $n \geqslant r \geqslant 1$,

$$A_q(n;r) \leqslant K_q(n;r) \leqslant \binom{r+q}{q-1}^{\lceil n/(r+1) \rceil}.$$

For $r = 1$ we may obtain improved upper-bounds (which are tight for $n = 3$).

Theorem 20. *Let* $\Sigma = \mathbb{Z}_q$ *be the alphabet,* $q \geqslant 2$, *and* $r = 1$. *Then for all* $3|n$,

$$A_q(n;r) \leqslant K_q(n;r) \leqslant \left[q + 2\binom{q}{2} + 2\binom{q}{3} \right]^{n/3}.$$

Here, for $q = 2$ we have $\binom{q}{3} = 0$.

In particular, for the binary case the last theorem provides an *asymptotic* upper bound of $2/3$ on the rate of $(n;1)$-LPC_∞ codes.

Thus far, we focused in this section and the previous on codes with a constant error-correction capability. This is motivated by the next corollary that shows that all other cases have asymptotic rate 0.

Corollary 21. *Let $\Sigma = \mathbb{Z}_q$ be the alphabet, $q \geqslant 2$ a constant. Let $\{C_i\}_{i\geqslant 1}$ be a sequence of codes, C_i being an $(n_i, M_i; r_i)_q$-LPC_∞ code, and $n_{i+1} > n_i$ for all $i \in \mathbb{N}$. If $r_i = \omega(1)$, i.e., $\limsup_{i\to\infty} r_i = \infty$, then the asymptotic rate of the family is*

$$\limsup_{i\to\infty} \frac{\log_q M_i}{n_i} = 0.$$

A. Specific code parameters

As we have already seen, the size of the error balls depends on the transmitted word and thus they are not all of the same size. The irregularity of the error balls restricts us from using classical ball-packing to obtain upper bounds on code size. For $r = 1$ and the binary case, we use the techniques of [10] (which employ a modified version of the ball-packing bound for the deletion channel) to obtain upper bounds on $A_2(n;1)$. Table I summarizes our upper bounds results together with the lower bound implied by the best codes we could find by computer search. At times our results are tight.

TABLE I
UPPER AND LOWER BOUNDS ON $A_2(n;1)$

n	Upper Bound	Lower Bound
3	4	4
4	8	8
5	12	12
6	16	16
7	30	28
8	46	42
9	64	64
10	116	104
11	178	157
12	256	246
13	450	388
14	696	594
15	1024	930
16	1750	1454

VI. CONCLUSION

In this work we initiate the study of ℓ_∞-limited permutation channels $\mathrm{LPC}_\infty(r)$ for worst case errors and general alphabets Σ. We present code constructions and upper bounds on code size. The majority of our results are for the case of $r = 1$. Despite significant efforts, our upper and lower bounds on code size are not tight and should be viewed as initial steps in a full understanding of $\mathrm{LPC}_\infty(r)$. For the case of binary codes with $r = 1$ we conjecture that the optimal asymptotic rate is $2/3$. This agrees with our upper bounds and our simulations up to block length $n = 24$. The optimal rate of codes for $\mathrm{LPC}_\infty(r)$ it left open and subject to future research.

REFERENCES

[1] G. Cohen, I. Honkala, S. Litsyn, and A. Lobstein, *Covering Codes*. North-Holland, 1997.

[2] M. Deza and H. Huang, "Metrics on permutations, a survey," *J. Comb. Inf. Sys. Sci.*, vol. 23, pp. 173–185, 1998.

[3] F. Farnoud, M. Schwartz, and J. Bruck, "Bounds for permutation rate-distortion," in *Proceedings of the 2014 IEEE International Symposium on Information Theory (ISIT2014), Honolulu, HI, USA*, Jun. 2014, pp. 6–10.

[4] K. A. S. Immink, *Coding Techniques for Digital Recorders*. Prentice-Hall, 1991.

[5] T. Kløve, "Lower bounds on the size of spheres of permutations under the Chebychev distance," *Designs, Codes and Cryptography*, vol. 59, no. 1-3, pp. 183–191, 2011.

[6] T. Kløve, T.-T. Lin, S.-C. Tsai, and W.-G. Tzeng, "Permutation arrays under the Chebyshev distance," *IEEE Trans. Inform. Theory*, vol. 56, no. 6, pp. 2611–2617, Jun. 2010.

[7] M. Kovačević and P. Popovski, "Zero-error capacity of a class of timing channels," Nov. 2013. [Online]. Available: http://arxiv.org/abs/1311.1339

[8] M. Kovačević and D. Vukobratović, "Multiset code for permutation channels," Jan. 2013. [Online]. Available: http://arxiv.org/abs/1301.7564

[9] V. Y. Krachkovsky, "Bounds on the zero-error capacity of the input-constrained bit-shift channel," *IEEE Trans. Inform. Theory*, vol. 40, no. 4, pp. 1240–1244, Jul. 1994.

[10] A. Kulkarni and N. Kiyavash, "Non-asymptotic upper bounds for deletion correcting codes," *IEEE Trans. Inform. Theory*, vol. 59, no. 8, pp. 5115–5130, Aug. 2013.

[11] D. H. Lehmer, "Permutations with strongly restricted displacements," in *Combinatorial Theory and its Applications II*, P. Erdős, A. Rényi, and V. T. Sós, Eds. Amsterdam, The Netherlands: North Holland, 1970, pp. 273–291.

[12] F. J. MacWilliams and N. J. A. Sloane, *The Theory of Error-Correcting Codes*. North-Holland, 1978.

[13] H. H. Permuter, P. Cuff, B. van Roy, and T. Weissman, "Capacity of the trapdoor channel with feedback," *IEEE Trans. Inform. Theory*, vol. 54, no. 7, pp. 3150–3165, Jul. 2008.

[14] M. Schwartz and J. Bruck, "On the capacity of the precision-resolution system," *IEEE Trans. Inform. Theory*, vol. 56, no. 3, pp. 1028–1037, 2010.

[15] M. Schwartz, "Efficiently computing the permanent and Hafnian of some banded Toeplitz matrices," *Linear Algebra and its Applications*, vol. 430, no. 4, pp. 1364–1374, Feb. 2009.

[16] M. Schwartz and I. Tamo, "Optimal permutation anticodes with the infinity norm via permanents of $(0,1)$-matrices," *J. Combin. Theory Ser. A*, vol. 118, pp. 1761–1774, 2011.

[17] S. Shamai (Shitz) and E. Zehavi, "Bounds on the capacity of the bit-shift magnetic recording channel," *IEEE Trans. Inform. Theory*, vol. 37, no. 3, pp. 863–872, Jul. 1991.

[18] I. Tamo and M. Schwartz, "Correcting limited-magnitude errors in the rank-modulation scheme," *IEEE Trans. Inform. Theory*, vol. 56, no. 6, pp. 2551–2560, Jun. 2010.

[19] ——, "On the labeling problem of permutation group codes for the infinity metric," *IEEE Trans. Inform. Theory*, vol. 58, no. 10, pp. 6595–6604, Oct. 2012.

[20] J. M. Walsh and S. Weber, "Capacity region of the permutation channel," in *Proc. 46-th Annual Allerton Conference on Communications, Control, and Computing, Monticello, IL, USA*, Sep. 2008, pp. 646–652.

[21] D. Wang, A. Mazumdar, and G. W. Wornell, "A rate distortion theory for permutation spaces," in *Proceedings of the 2013 IEEE International Symposium on Information Theory (ISIT2013), Istanbul, Turkey*, Jul. 2013, pp. 2562–2566.

[22] H. Zhou, M. Schwartz, A. Jiang, and J. Bruck, "Systematic error-correcting codes for rank modulation," *IEEE Trans. Inform. Theory*, vol. 61, no. 1, pp. 17–32, Jan. 2015.