Generalized Sphere Packing Bound: Applications

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Abstract—**In this paper we study a generalization of the sphere packing bound for channels that are not regular (the size of balls with a fixed radius is not necessarily the same). Our motivation to tackle this problem is originated by a recent work by Kulkarni and Kiyavash who introduced a method, based upon tools from hypergraph theory, to calculate explicit upper bounds on the cardinalities of deletion-correcting codes. Under their setup, the deletion channel is represented by a hypergraph such that every deletion ball is a hyperedge. Since every code is a matching in the hypergraph, an upper bound on the codes is given by an upper bound on the largest matching in a hypergraph. This bound, called here the *generalized sphere packing bound*, can be found by the solution of a linear programming problem.**

We similarly study and analyze specific examples of error channels. We start with the Z channel and show how to exactly find the generalized sphere packing bound for this setup. Next studied is the non-binary limited magnitude channel both for symmetric and asymmetric errors. We focus on the case of single error and derive upper bounds on the generalized sphere packing bound in this channel. We follow up on the deletion case, which was the original motivation of the work by Kulkarni and Kiyavash, and show how to improve upon their upper bounds for the single deletion case. Finally, we apply this method for projective spaces and find its generalized sphere packing bound for the single-error case.

I. Introduction

In coding theory, there are many channels, such as the deletion and Z channels, which are not regular, that is, the size of balls with fixed radius is not necessarily the same. Usually, the study of these channels is more challenging since many lower and upper bounds cannot be applied on these channels since they require the channel to be regular. One fundamental example for such a bound is the ***sphere packing bound***, which asserts that for regular channels an upper bound on a code correcting t errors is the ratio between the size of the space and a size of a ball of radius t. In this paper, we study how to generalize the sphere packing bound in paradigms where the channels are not regular.

The approach we take in generalizing the sphere packing bound is inspired by a recent study by Kulrkani and Kiyavash. They presented a scheme, based upon tools from hypergraph theory, which gives non-asymptotic upper bounds for deletion-correcting codes. In this paper, we use results on the generalization of their method from [6] in order to provide specific examples on its application to channels that are not regular. These examples include the Z channel, non-binary channels with limited magnitude errors (symmetric and asymmetric), deletion channel, and finally, projective spaces. When possible in these examples, we compare the bounds we receive with the state-of-the-art ones.

Let $\mathcal{H} = (X, \mathcal{E})$ be a hypergraph, where $X = \{x_1, \ldots, x_n\}$ is its vertex set and $\mathcal{E} = \{E_1, \ldots, E_m\}$ is its hyperedge set. Let A be the $n \times m$ incidence matrix of $\mathcal{H}$, so $A(i, j) = 1$ if $x_i \in E_j$. A ***matching*** in $\mathcal{H}$ is a collection of pairwise disjoint hyperedges and the ***matching number*** of $\mathcal{H}$, denoted by $\nu(\mathcal{H})$, is the size of the largest matching. The dual definition of a matching is a ***transversal***, which is a subset $T \subseteq X$ that intersects every hyperedge in $\mathcal{E}$. The ***transversal number*** of $\mathcal{H}$, denoted by $\tau(\mathcal{H})$, is the size of the smallest transversal. It is easy to verify that the size of any transversal gives an upper bound on the matching number and thus $\nu(\mathcal{H}) \leqslant \tau(\mathcal{H})$. Every transversal can be represented by a binary vector $\boldsymbol{w} \in \{0, 1\}^n$ which needs to satisfy $A^T \cdot \boldsymbol{w} \geqslant \mathbf{1}$. However, if the vector $\boldsymbol{w}$ can have values over $\mathbb{R}_+$ and still satisfies the last inequality, then it is called a ***fractional transversal***. Under this setup, it is known that $\nu(\mathcal{H}) \leqslant \tau^*(\mathcal{H}) \leqslant \tau(\mathcal{H})$, where $\tau^*(\mathcal{H})$ is the linear programming relaxation of $\tau(\mathcal{H})$, defined as

$$\tau^*(\mathcal{H}) = \min \left\{ \sum_{i=1}^{n} w_i : A^T \cdot \boldsymbol{w} \geqslant \mathbf{1}, \boldsymbol{w} \in \mathbb{R}_+^n \right\}. \quad (1)$$

Let $\mathcal{G} = (X, E)$ be a directed graph which describes an error channel. The vertex set X is the set of all possible transmitted words, and the edges set E consists of all pairs of vertices at distance one. The distance between $x, y \in X$, is the path metric in $\mathcal{G}$ and is denoted by $d(x, y)$. Note that since the graph is directed, it is possible to have $d(x, y) \neq d(y, x)$. For every $x \in X$, its radius-r ball is the set $B_r(x) \triangleq \{y \in X \mid d(x, y) \leqslant r\}$ and its degree is $\deg_r(x) = |B_r(x)|$. A code $\mathcal{C} \subseteq X$ is said to be an r-error-correcting code if for all $x, y \in \mathcal{C}$, $B_r(x) \cap B_r(y) = \emptyset$. We let $A_{\mathcal{G}}(n, r)$ be the largest cardinality of an r-error-correcting code in $\mathcal{G}$ of length n. Given some positive integer r, the graph $\mathcal{G}$ is associated with a hypergraph $\mathcal{H}(\mathcal{G}, r) = (X_r, \mathcal{E}_r)$ where $X_r = X$ and $\mathcal{E}_r = \{B_r(x) \mid x \in X\}$. Observing that every r-error-correcting code $\mathcal{C} \subseteq X$ is a matching in $\mathcal{H}(\mathcal{G}, r)$, the following upper bound on $A_{\mathcal{G}}(n, r)$ was verified in [8],

$$A_{\mathcal{G}}(n, r) \leqslant \tau^*(\mathcal{H}(\mathcal{G}, r)). \quad (2)$$

One of the properties shown in [6], asserts that for a regular and symmetric graph $\mathcal{G}$ the bound $\tau^*(\mathcal{H}(\mathcal{G}, r))$ in (2) coincides with the sphere packing bound, and therefore the bound $\tau^*(\mathcal{H}(\mathcal{G}, r))$ is called the ***generalized sphere packing bound***.

The expression $\tau^*(\mathcal{H}(\mathcal{G}, r))$ provides an explicit upper bound on $A_{\mathcal{G}}(n, r)$. However, it may still be a hard problem to calculate this value since it requires the solution of a linear programming problem that can have an exponential number of variables and constraints. Clearly, one would aspire to find this exact value, but if this is not possible to accomplish, it is still valuable to give an upper bound on $\tau^*(\mathcal{H}(\mathcal{G}, r))$, which is an upper bound on $A_{\mathcal{G}}(n, r)$ as well. Such an upper bound will be given by finding any fractional transversal and the goal will be to find one with small weight.

The rest of the paper is organized as follows. In Section II we review the definitions and tools from [6] and [8] required to solve the problems in this paper. In Section III, we study the Z channel. Our main contribution here is finding a method to calculate the general sphere packing bound for all radii. In Section IV we carry a similar task for the limited-magnitude channel with symmetric and asymmetric errors. We focus only

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on the single error case in both scenarios and find fractional transversals and corresponding upper bounds. Section V follows upon the original work of [8], improving the bounds derived therein for the deletion channel (for the case of a single deletion). Section VI studies bounds on projective spaces and gives an optimal solution for the radius-one case. Due to the lack of space, some of the proofs in the paper are omitted.

II. Definitions and Preliminaries

Let $\mathcal{G} = (X, E)$, where $X = \{x_1, \ldots, x_n\}$, be a directed graph describing an error channel and $\mathcal{H}(\mathcal{G}, r) = (X_r, \mathcal{E}_r)$ is its associated hypergraph. Our main goal in this work is to calculate the generalized sphere packing bound stated in (2) for the examples mentioned in the Introduction. This may not be an easy task since the linear programming problem (1) calculating the value $\tau^*(\mathcal{H}(\mathcal{G}, r))$ may have an exponential number of variables and constraints. However, some tools, summarized as follows and given in details in [6], can significantly help in calculating this bound.

The graph $\mathcal{G}$ is called ***monotone*** if for every $r \geqslant 1$, $x \in X$ and $y \in B_r(x)$, $\deg_r(y) \leqslant \deg_r(x)$. This property is helpful in giving an explicit construction of a fractional transversal $\boldsymbol{w} = (w_1, \ldots, w_n)$ where $w_i = \frac{1}{\deg_r(x_i)}$, which provides the following upper bound

$$A_{\mathcal{G}}(n, 2r+1) \leqslant \sum_{i=1}^{n} w_i = \sum_{i=1}^{n} \frac{1}{\deg_r(x_i)}.$$

This bound is called the ***monotonicity upper bound*** and is denoted by $MB(\mathcal{G}, r)$.

Lastly, we review a method from [6] that can significantly reduce the complexity in calculating the linear programming problem of the value $\tau^*(\mathcal{H}(\mathcal{G}, r))$. Let us first remind some tools derived from properties on automorphisms of graphs. Let $G = (X, E)$ be a directed graph with n vertices. An ***automorphism*** of a graph $G = (X, E)$ is a permutation $\pi : X \to X$ that preserves adjacency, i.e., for all $(x, y) \in X \times X$, $(x, y) \in E$ if and only if $(\pi(x), \pi(y)) \in E$. Let $\mathbb{S}_n$, where $|X| = n$, be the set of all permutations of n elements, and $Aut(G)$ be the set of all automorphisms of G. It is known that $Aut(G)$ is a subgroup of $\mathbb{S}_n$.

Every subgroup H of $Aut(G)$ induces a relation R_H, which is an equivalence order, on X where $(x, y) \in R_H$ if and only if there exists $\pi \in H$ such that $\pi(x) = y$. Let $X_1, \ldots, X_{n_H}$ be a partition of X into n_H equivalence classes according to H. Let A_H be an $n_H \times n_H$ adjacency matrix corresponding to the subgroup H, such that for $1 \leqslant i, j \leqslant n_H$,

$$A_H(i, j) = \frac{|\{(x, y) : y \in B_r(x) \cap X_j\}|}{|X_i|}.$$

The main advantage in this approach is that vertices belonging to the same equivalence class can be assigned with the same weight when solving the linear programming in (1). This simplification can significantly reduce the number of variables and constraints in the linear programming. This property is stated in the next theorem from [6].

Theorem 1. *Let H be a subgroup of $Aut(\mathcal{G})$ and $\mathcal{X}_H(\mathcal{G}) = \{X_1, \ldots, X_{n_H}\}$ is its partition of X into n_H equivalence classes. Then, the generalized sphere packing bound $\tau^*(\mathcal{H}(\mathcal{G}, r))$ from (1) becomes*

$$\tau^*(\mathcal{H}(\mathcal{G}, r)) = \min\left\{ \sum_{i=1}^{n_H} |X_i| \cdot w_i : A_H^T \cdot \boldsymbol{w} \geqslant 1, \boldsymbol{w} \in \mathbb{R}_+^{n_H} \right\}. \quad (3)$$

III. The Z Channel

We let the graph of the Z channel be $\mathcal{G}_Z = (X_Z, E_Z)$, where $X_Z = \{0,1\}^n$ and $E_Z = \{(\boldsymbol{x}, \boldsymbol{y}) : \boldsymbol{x}, \boldsymbol{y} \in \{0,1\}^n, \boldsymbol{x} \geqslant \boldsymbol{y}, w_H(\boldsymbol{x}) = w_H(\boldsymbol{y}) + 1\}$, and $w_H(\boldsymbol{x})$ denotes the Hamming weight of $\boldsymbol{x}$. So errors can change a 1 to 0 but not vice versa. Let r be some fixed positive integer. For every $\boldsymbol{x} \in \{0,1\}^n$,

$$B_{Z,r}(\boldsymbol{x}) = \{\boldsymbol{y} \in \{0,1\}^n \ : \ \boldsymbol{x} \geqslant \boldsymbol{y}, w_H(\boldsymbol{x}) - w_H(\boldsymbol{y}) \leqslant r\},$$

and $\deg_{Z,r}(\boldsymbol{x}) = \sum_{i=0}^{r} \binom{w_H(\boldsymbol{x})}{i}$. The hypergraph for this channel is $\mathcal{H}(\mathcal{G}_Z, r) = (X_{Z,r}, \mathcal{E}_{Z,r})$, where $X_{Z,r} = \{0,1\}^n$ and $\mathcal{E}_{Z,r} = \{B_{Z,r}(\boldsymbol{x}) \ : \ \boldsymbol{x} \in \{0,1\}^n\}$. In order to find the value $\tau^*(\mathcal{H}(\mathcal{G}_Z, r))$, it is required to solve a linear programming with 2^n variables and 2^n constraints. However, according to the automorphism scheme from Theorem 1, it can be reduced to only $n+1$ variables and constraints as follows

$$\tau^*(\mathcal{H}(\mathcal{G}_Z, r)) = \min\left\{ \sum_{\ell=0}^{n} \binom{n}{\ell} w_\ell : \sum_{i=0}^{\min\{\ell, r\}} \binom{\ell}{i} w_{\ell-i} \geqslant 1, 0 \leqslant \ell \leqslant n \right\}. \quad (4)$$

The goal of this section is to solve the linear programming problem in (4) by finding the appropriate fractional transversal that calculates this value for all $r \leqslant 20$. We note that the special case of $r = 1$ is studied in our companion paper [6].

Theorem 2. *For all $r \leqslant 20$, the optimal fractional transversal $\boldsymbol{w}^*$ which solves the linear programming in (4) is given by the following recursive formula*

$$w_0^* = 1, w_n^* = w_{n-1}^* = \cdots = w_{n-r+1}^* = 0, \quad (5)$$

$$w_k^* = (1 - \sum_{i=1}^{r} w_{k+i}^* \binom{k+r}{r-i}) / \binom{k+r}{r}, \forall 1 \leqslant k \leqslant n-r.$$

Proof: First, we introduce an explicit formula for $\boldsymbol{w}^*$ by $w_0^* = 1$ and for $1 \leqslant k \leqslant n$,

$$w_k^* = r!k! \sum_{m=r+k}^{n} \frac{D_{m-k-1}}{m!}, \quad (6)$$

where D_i is a sequence independent of n and is given by

$$D_0 = D_1 = \cdots = D_{r-2} = 0, D_{r-1} = 1,$$

$$\frac{D_i}{r!} + \frac{D_{i-1}}{(r-1)!} + \cdots + \frac{D_{i-r}}{0!} = 0 \qquad \forall i \geqslant r.$$

The equivalence of the two formulas comes from a couple of inductions (see [5] for more details.) The proof consists of two parts, feasibility (transversal property) and the optimality of our choice. Here, we overview the feasibility proof using the equivalent formula, but we leave the details along with the optimality proof in the longer version of the paper [5].

The recursive formula for $\boldsymbol{w}^*$ gives us the transversal inequality constraints. So, it suffices to show the non-negativity of $\boldsymbol{w}^*$ to prove its feasibility.

A simple induction on m gives $|D_m| \leqslant (2r)^{m-r+1}$ for all $r \in \mathbb{N}$. As a result, the growth of D_m is no more than an exponential function of $2r$ and moreover, $\lim_{n\to\infty} w_k^*$ exists. Now, the main idea is to look at the first terms in (6) and show that they cannot be cancelled out by the remaining terms.

Let us first verify the statement $w_k^* > 0$ for $k \geqslant 3r-1$.

$$w_k^* = r!k! \sum_{m=r+k}^{n} \frac{D_{m-k-1}}{m!} = r!k! (\frac{1}{(r+k)!} + \sum_{m=r+k+1}^{n} \frac{D_{m-k-1}}{m!})$$

$$\geqslant \frac{r!k!}{(r+k)!} (1 - \sum_{m=r+k+1}^{n} \frac{(2r)^{m-r-k}}{(4r)^{m-r-k}}) = \frac{r!k!}{(r+k)!} 2^{-(n-k-r)},$$

which is a positive value. In general, we do not have the proof for the case when $k < 3r-1$. However, for any fixed r we can research the feasibility in the following fashion: Given $k < 3r-1$, we look for a number n_k such that

$$\sum_{m=r+k}^{n_k} \frac{D_{m-k-1}}{m!} \geqslant \frac{1}{(2r)^{r+k}}\left(e^{2r} - \sum_{m=0}^{n_k} \frac{(2r)^m}{m!}\right),$$

which means for all $n > n_k$ we have

$$\frac{w_k^*}{r!k!} = \sum_{m=r+k}^{n_k} \frac{D_{m-k-1}}{m!} + \sum_{m=n_k+1}^{n} \frac{D_{m-k-1}}{m!} > \sum_{m=r+k}^{n_k} \frac{D_{m-k-1}}{m!}$$

$$-\frac{1}{(2r)^{k+r}} \sum_{m=n_k+1}^{\infty} \frac{(2r)^m}{m!} = \sum_{m=r+k}^{n_k} \frac{D_{m-k-1}}{m!} - \frac{e^{2r} - \sum_{m=0}^{n_k} \frac{(2r)^m}{m!}}{(2r)^{r+k}} \geqslant 0;$$

And then we check the values of w_k^* for the finite set of $k < 3r-1$ and $n \leqslant n_k$. We mention that if $w_k^* \geqslant 0$ for all n, then the number n_k exists since $\lim_{n_k \to \infty} \sum_{m=0}^{n_k} \frac{(2r)^m}{m!} = e^{2r}$. As an example, when $r = 2$ we have $n_1 = n_2 = 6$, and $n_3 = n_4 = 7$. Using the above approach, we have verified the feasibility for all $r \leqslant 20$. Our calculations also show that $n_k \leqslant 4r - 1$ for all $n \leqslant 20$. ■

Lastly, we note that it is possible to verify the optimality of the weight assignment from (5) for arbitrary r using the method in Theorem 2.

IV. Limited Magnitude Channels

We turn in this section to extend the Z channel for the non-binary case. In this setup, every symbol can have q values, $0, 1, \ldots, q-1$ and we denote $[q] = \{0, 1, \ldots, q-1\}$. We study the limited magnitude model and focus solely on the single error setup which is carried for two cases. Namely, the error can be asymmetric or symmetric.

In the asymmetric non-binary channel, the value of every symbol can only decrease, and we only consider the case where the values can decrease by one. The corresponding graph is $\mathcal{G}_{A,q} = (X_{A,q}, E_{A,q})$, where $X_{A,q} = [q]^n$ and

$$E_{A,q} = \left\{ (\boldsymbol{x}, \boldsymbol{y}) : \boldsymbol{x}, \boldsymbol{y} \in [q]^n, \boldsymbol{x} \geqslant \boldsymbol{y}, \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i + 1 \right\}.$$

Given some $\boldsymbol{x} \in [q]^n$, its ball of radius one is described by the set $B_{A,q,1}(\boldsymbol{x}) = \{\boldsymbol{y} \in [q]^n : \boldsymbol{x} \geqslant \boldsymbol{y}, \sum_{i=1}^n x_i \leqslant \sum_{i=1}^n y_i + 1\}$, and $\deg_{A,q,1}(\boldsymbol{x}) = w_H(\boldsymbol{x}) + 1$. The hypergraph in this case is $\mathcal{H}(\mathcal{G}_{A,q}, 1) = (X_{A,q,1}, \mathcal{E}_{A,q,1})$, where $X_{A,q,1} = [q]^n$ and $\mathcal{E}_{A,q,1} = \{B_{A,q,1}(\boldsymbol{x}) : \boldsymbol{x} \in [q]^n\}$.

According to the above definitions it is immediate to verify that for all $\boldsymbol{y} \in B_{A,q,1}(\boldsymbol{x})$, $w_H(\boldsymbol{y}) \leqslant w_H(\boldsymbol{x})$ and thus the graph $\mathcal{G}_{A,q}$ is monotone. In the next lemma we state the monotonicity upper bound for radius one.

Lemma 3. *The monotonicity upper bound of the graph $\mathcal{G}_{A,q}$ for $r = 1$ is given by*

$$MB(\mathcal{G}_{A,q}, 1) = \frac{q^{n+1}}{(q-1)(n+1)}.$$

The linear programming problem from (1) becomes

$$\tau^*(\mathcal{H}(\mathcal{G}_{A,q,1})) = \min \left\{ \sum_{\boldsymbol{x} \in [q]^n} w_{\boldsymbol{x}} \ : \ \sum_{\boldsymbol{y} \in B_{A,q,1}(\boldsymbol{x})} w_{\boldsymbol{y}} \geqslant 1 \right\}.$$

However, it can be significantly simplified according to the automorphism tools summarized in Theorem 1. For every permutation $\sigma \in \mathbb{S}_n$ we define a permutation $\pi_\sigma = [q]^n \to [q]^n$ such that for all $\boldsymbol{x} \in [q]^n$, $(\pi_\sigma(\boldsymbol{x}))_i = x_{\sigma(i)}$. Hence, the set $H_A = \{\pi_\sigma : \sigma \in \mathbb{S}_n\}$ is a subgroup of $Aut(\mathcal{G}_{A,q})$ which partitions $[q]^n$ into $n^* = \binom{n+q-1}{q-1}$ equivalence classes

$$\mathcal{X} = \left\{ X_{\boldsymbol{i}} \ : \ \boldsymbol{i} = (i_0, \ldots, i_{q-1}) \geqslant \boldsymbol{0}, \sum_{j=0}^{q-1} i_j = n \right\},$$

where $X_{\boldsymbol{i}} = \{\boldsymbol{x} \in [q]^n : \boldsymbol{x}^{-1}(j) = i_j, 0 \leqslant j \leqslant q-1\}$, and $\boldsymbol{x}^{-1}(j) = |\{1 \leqslant k \leqslant n : x_k = j\}|$. We denote the set $\mathbf{I}$ to be $\mathbf{I} = \{\boldsymbol{i} \ : \ \boldsymbol{i} = (i_0, \ldots, i_{q-1}) \geqslant \boldsymbol{0}, \sum_{j=0}^{q-1} i_j = n\}$ and define an $n^* \times n^*$ matrix A_H such that its entries are the vectors $(\boldsymbol{i}, \boldsymbol{j}) \in \mathbf{I} \times \mathbf{I}$. We assign the values $A_H(\boldsymbol{i}, \boldsymbol{i}) = 1$ and $A_H(\boldsymbol{i}, \boldsymbol{j}) = i_k$ if there exists $1 \leqslant k \leqslant q-1$ such that $j_k = i_k - 1$ and $j_{k-1} = i_{k-1} + 1$ and for all $\ell \in [q] \setminus \{k, k-1\}$, $j_\ell = i_\ell$. All other values in the matrix A_H are assigned with the value 0. Thus, we can state the following theorem.

Theorem 4. *The generalized sphere packing for $\mathcal{H}(\mathcal{G}_{A,q,1})$ is given by*

$$\tau^*(\mathcal{H}(\mathcal{G}_{A,q,1})) = \min \left\{ \sum_{\boldsymbol{i} \in I} |X_{\boldsymbol{i}}| \cdot w_{\boldsymbol{i}} : A_H^T \cdot \boldsymbol{w} \geqslant 1, \boldsymbol{w} = (w_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathbf{I}} \in \mathbb{R}_+^{n^*} \right\}.$$

We finish this section by showing an improvement upon the suboptimal monotonicity upper bound from Lemma 3. In the fractional transversal notation of Theorem 4, if one applied the monotonicity upper bound, then the fractional transversal assignment would be $w_{\boldsymbol{i}} = 1/(n - i_0 + 1)$ for $\boldsymbol{i} \in \mathbf{I}$. However, under this assignment almost all of the constraints hold with strict inequality. We show that it is possible to reduce the weights in this assignment without violating the constraints.

Theorem 5. *The vector $\boldsymbol{w} = (w_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathbf{I}}$ given by*

$$w_{\boldsymbol{i}} = \frac{1}{n - i_0 + 1 + \frac{i_1 - 1}{2(n - i_0)}},$$

if $i_0 \neq n$ and otherwise $w_{\boldsymbol{i}} = 1$ is a fractional transversal for $\tau^*(\mathcal{H}(\mathcal{G}_{A,q,1}))$ as stated in Theorem 4.

Table I summarizes the upper bounds results we derived in this section for $q = 3$. The first column is the monotonicity upper bound. The second column is the improvement in Theorem 5 over the monotonicity upper bound. The last column is the value of the generalized sphere packing bound from Theorem 4, which we solved numerically.

TABLE I
Asymmetric errors: upper bounds comparison for $q = 3$

n	MB	Theorem 5	GSPB
5	60	60	55
6	156	154	144
7	410	402	381
8	1093	1071	1021
9	2952	2888	2770
10	8052	7877	7591

We omit the details and results of the symmetric channel due to the lack of space.

V. The Deletion Channel

In this section we shift our attention to the deletion channel, which was the original usage of the generalized sphere packing bound in [8]. We will only focus on the single deletion case and our main result will be an explicit expression of a fractional transversal which improves upon the one from [8].

We first introduce the graph for the deletion channel. However, note that this graph is different than the ones studied so far. Specifically, a length n vector which suffers a single deletion will result with a vector of length $n-1$. To accommodate this structure, the vertices in the graph are defined to be both vectors of length n and $n-1$, so the graph is $\mathcal{G}_D = (X_D, E_D)$, where $X_D = \{0,1\}^n \cup \{0,1\}^{n-1}$ and

$$E_D = \{(\boldsymbol{x},\boldsymbol{y}) \in \{0,1\}^n \times \{0,1\}^{n-1} : \\ \boldsymbol{y} = (x_1,\ldots,x_i,x_{i+2},\ldots,x_n) \text{ for some } 1 \leqslant i \leqslant n\}.$$

For any $\boldsymbol{x} \in \{0,1\}^n$, its radius one ball is the set $B_{D,1}(\boldsymbol{x}) = \{\boldsymbol{y} \in \{0,1\}^{n-1} : (\boldsymbol{x},\boldsymbol{y}) \in E_D\}$, and for $\boldsymbol{x} \in \{0,1\}^{n-1}$, $B_{D,1}(\boldsymbol{x}) = \emptyset$. However, since the length-n vectors do not participate in the balls we can eliminate them in the hypergraph construction. Thus the hypergraph here is $\mathcal{H}(\mathcal{G}_D,1) = (X_{D,1},\mathcal{E}_{D,1})$, where $X_{D,1} = \{0,1\}^{n-1}$ and $\mathcal{E}_{D,1} = \{B_{D,1}(\boldsymbol{x}) : \boldsymbol{x} \in \{0,1\}^n\}$. The generalized sphere packing bound in this setup becomes

$$\tau^*(\mathcal{H}(\mathcal{G}_D,1)) = \min\left\{ \sum_{z\in\{0,1\}^{n-1}} w_z : \sum_{y\in B_{D,1}(x)} w_y \geqslant 1, \forall \boldsymbol{x} \in \{0,1\}^n \right\}. \tag{7}$$

For a vector $\boldsymbol{x} \in \{0,1\}^n$, we denote by $\rho(\boldsymbol{x})$ the number of runs in $\boldsymbol{x}$. For example, if $\boldsymbol{x} = 001010010$, then $\rho(\boldsymbol{x}) = 7$. It is easily verified that for $\boldsymbol{x} \in \{0,1\}^n$, $\deg_{D,1}(\boldsymbol{x}) = \rho(\boldsymbol{x})$, [8]. It is also known that the number of length-n vectors with $1 \leqslant \rho \leqslant n$ runs is $2\binom{n-1}{\rho-1}$.

In the structure of $\mathcal{G}_D$, it is not possible to indicate whether the graph $\mathcal{G}_D$ satisfies the monotonicity property. However, there is still a similar property to the monotonicity one. Namely, for every $\boldsymbol{y} \in B_{D,1}(\boldsymbol{x})$, where $\boldsymbol{x} \in \{0,1\}^n$, $\rho(\boldsymbol{y}) \leqslant \rho(\boldsymbol{x}) = \deg_{D,1}(\boldsymbol{x})$. This property was established in [8] and thus a choice of a fractional transversal $(w_{\boldsymbol{x}})_{\boldsymbol{x}\in\{0,1\}^{n-1}}$, was given by $w_{\boldsymbol{x}} = 1/\rho(\boldsymbol{x})$. The corresponding upper bound, which we call here the monotonicity upper bound, was calculated in [8] to be $\frac{2^n-2}{n-1}$.

It is possible to verify that for the last fractional transversal many of the constraints in the linear programming in (7) hold with strong inequality, which implies that a better one could be found. This will be the focus in the rest of this section.

For a vector $\boldsymbol{x}$, let $\mu(\boldsymbol{x})$ be the number of middle runs (i.e., not on the edges) of length 1 in $\boldsymbol{x}$. We call these runs *middle-1-runs*. For example, for $\boldsymbol{x} = 001010010$, $\mu(\boldsymbol{x}) = 4$. First notice that if $\rho(\boldsymbol{x}) \geqslant 2$ then $0 \leqslant \mu(\boldsymbol{x}) \leqslant \rho(\boldsymbol{x}) - 2$. Let $N_n(\rho,\mu)$ denote the number of vectors of length n with ρ runs and μ middle-1-runs. For $\rho = 1$ and $\mu = 0$, we have $N_n(1,0) = 2$. For $2 \leqslant \rho \leqslant n$ and $0 \leqslant \mu \leqslant \rho - 2$, the value of $N_n(\rho,\mu)$ is stated in the next lemma. For all other values of ρ and μ the value of $N_n(\rho,\mu)$ is zero.

Lemma 6. *For $2 \leqslant \rho \leqslant n$ and $0 \leqslant \mu \leqslant \rho - 2$,*

$$N_n(\rho,\mu) = 2\binom{\rho-2}{\mu}\binom{n-\rho+1}{\rho-\mu-1}.$$

Next, the main result in this section is proved.

Theorem 7. *The vector $\boldsymbol{w} = (w_{\boldsymbol{x}})_{\boldsymbol{x}\in\{0,1\}^{n-1}}$ defined by*

$$w_{\boldsymbol{x}} = \begin{cases} \frac{1}{\rho(\boldsymbol{x})} & \text{if } \mu(\boldsymbol{x}) \leqslant 1 \\ \frac{1}{\rho(\boldsymbol{x})}\left(1 - \frac{\mu(\boldsymbol{x})}{\rho(\boldsymbol{x})^2}\right) & \text{otherwise} \end{cases}$$

is a fractional transversal.

Proof: Let $\boldsymbol{x}$ be a length-n binary vector with ρ runs and μ middle-1-runs. We need to show that $\sum_{y\in B_{D,1}(x)} w_{\boldsymbol{y}} \geqslant 1$. It can be verified that this claim holds for $\rho = 1,2,3$ or $\mu = 0,1$ and thus we assume for the rest of the proof that $\rho \geqslant 4$ and $\mu \geqslant 2$. Note that for a fixed ρ, $w_{\boldsymbol{x}}$ is decreasing when μ increases.

If a vector $\boldsymbol{y} \in B_{D,1}(\boldsymbol{x})$ is received by deleting a middle-1-run bit then $\rho(\boldsymbol{y}) = \rho - 2$ and $\mu - 3 \leqslant \mu(\boldsymbol{y}) \leqslant \mu - 1$. Otherwise, $\rho(\boldsymbol{y}) = \rho$ and $\mu(\boldsymbol{y}) \leqslant \mu + 1$ or $\rho(\boldsymbol{y}) = \rho - 1$ and $\mu(\boldsymbol{y}) \leqslant \mu$, however, the worst case in terms of the value of $w_{\boldsymbol{y}}$ is achieved for $\rho(\boldsymbol{y}) = \rho$ and $\mu(\boldsymbol{y}) = \mu + 1$. Therefore,

$$\sum_{y\in B_{D,1}(x)} w_y \geqslant \frac{\mu}{\rho-2}\left(1 - \frac{\mu-1}{(\rho-2)^2}\right) + \frac{(\rho-\mu)}{\rho}\left(1 - \frac{\mu+1}{\rho^2}\right)$$

$$= 1 + \frac{2\mu}{\rho(\rho-2)} - \frac{\mu(\mu-1)}{(\rho-2)^3} - \frac{\mu+1}{\rho^2} + \frac{\mu(\mu+1)}{\rho^3},$$

and thus it is enough to show that for $4 \leqslant \rho, 2 \leqslant \mu \leqslant \rho - 2$,

$$\frac{2\mu}{\rho(\rho-2)} - \frac{\mu(\mu-1)}{(\rho-2)^3} - \frac{\mu+1}{\rho^2} + \frac{\mu(\mu+1)}{\rho^3} \geqslant 0,$$

or

$$\frac{2}{\rho(\rho-2)} - \frac{1}{\rho^2} \geqslant \frac{\mu-1}{(\rho-2)^3} + \frac{1}{\mu\rho^2} - \frac{\mu+1}{\rho^3}.$$

The function $f(\mu) = \frac{\mu-1}{(\rho-2)^3} + \frac{1}{\mu\rho^2} - \frac{\mu+1}{\rho^3}$ in the range $2 \leqslant \mu \leqslant \rho - 2$ is maximized either when $\mu = 2$ or $\mu = \rho - 2$ and thus we need to show that

$$\frac{2}{\rho(\rho-2)} - \frac{1}{\rho^2} \geqslant \frac{1}{(\rho-2)^3} + \frac{1}{2\rho^2} - \frac{3}{\rho^3},$$

and

$$\frac{2}{\rho(\rho-2)} - \frac{1}{\rho^2} \geqslant \frac{\rho-3}{(\rho-2)^3} + \frac{1}{(\rho-2)\rho^2} - \frac{\rho-1}{\rho^3},$$

which holds for all $\rho \geqslant 4$. ■

For a vector $\boldsymbol{x}$ with ρ runs and μ middle-1-runs, we denote its weight by $w(\rho,\mu)$. From Lemma 6 and Theorem 7 we conclude with the following upper bound on $\tau^*(\mathcal{H}(\mathcal{G}_D,1))$.

Theorem 8. *The value $\tau^*(\mathcal{H}(\mathcal{G}_D,1))$ satisfies*

$$\tau^*(\mathcal{H}(\mathcal{G}_D,1)) \leqslant 2 + \sum_{\rho=2}^{n-1}\sum_{\mu=0}^{\rho-2} N_{n-1}(\rho,\mu)w(\rho,\mu).$$

Table II summarizes the results in this section. MB corresponds to the equivalent of the monotonicity upper bound from [8]. The second column is our upper bound from Theorem 8. The column titled GSPB [8] is the exact value of $\tau^*(\mathcal{H}(\mathcal{G}_D,1))$ from (7), which this linear programming problem was numerically solved in [8] for $n \leqslant 14$. Since this linear programming has a large number of constraints and variables it is numerically hard to solve it for larger values of n. The last column LB corresponds to the lower bound, which is the best known construction of single-deletion codes from [9].

TABLE II
DELETION CHANNEL COMPARISON

n	MB [8]	Theorem 8	GSPB [8]	LB [9]
5	7	7	6	6
6	12	12	10	10
7	21	20	17	16
8	36	35	30	30
9	63	61	53	52
10	113	109	96	94
11	204	197	175	172
12	372	358	321	316
13	682	657	593	586
14	1260	1212	1104	1096
15	2340	2251	?	2048
16	4368	4202	?	3856
17	8191	7882	?	7286
18	15420	14845	?	13798
19	29127	28059	?	26216
20	55188	53202	?	49940

VI. Projective Spaces

In this section, we explain an example where there is no monotonocity property, yet we benefit from the graph automorphisms and we simplify the linear programming again. Koetter and Kschischang [7] modeled codes as subsets of projective space $\mathbb{F}_q^n$, the set of linear subspaces of $\mathbb{F}_q^n$, or of Grassmann space $\mathcal{G}(n,k)$, the subset of linear subspaces of $\mathbb{F}_q^n$ having dimension k. Subsets of $\mathbb{F}_q^n$ are called *projective codes* and similarly to previous sections, it is desired to select elements with large distance from each other.

Let us first introduce the graph $\mathcal{G}_P = (X_P, E_P)$ for projective codes, where X_P is the set of all linear subspaces in $\mathbb{F}_q^n$ and

$$E_P = \{\{x,y\} : x \subset y \text{ or } y \subset x, \text{ and } |\dim(x) - \dim(y)| = 1\},$$

and using the path distance $d_P(x,y)$ defined on the graph $\mathcal{G}_P$ we define

$$\mathcal{B}_{P,r}(x) = \{y \in \mathcal{X}_P : d_P(x,y) \leqslant r\}.$$

The corresponding hypergraph is $\mathcal{H}(\mathcal{G}_P, r) = (X_{P,r}, \mathcal{E}_{P,r})$, such that $X_{P,r} = X_P$ and $\mathcal{E}_{P,r} = \{\mathcal{B}_{P,r}(x) : x \in X_P\}$. The generalized sphere packing bound becomes

$$\tau^*(\mathcal{H}(\mathcal{G}_P, r)) = \min\left\{\sum_{x\in X_P} w(x) : \forall x \in \mathcal{X}_P, \sum_{y\in\mathcal{B}_{P,r}(x)} w_y \geqslant 1, w_x \geqslant 0\right\}.$$

Assume x_1 and x_2 are elements in X_P of the same dimension k. There exists an injective linear transform $\mathcal{T} : \mathbb{F}_q^n \to \mathbb{F}_q^n$ mapping the basis of x_1 into a basis for x_2. Note that $x \subseteq y$ if and only if $\mathcal{T}(x) \subseteq \mathcal{T}(y)$. Hence, all such linear transforms are automorphisms on $\mathcal{G}_P$, which means for any $x_1, x_2 \in X_P$ of the same dimensions, there exists an automorphism mapping between them. Therefore, they lie in a same equivalence class. So we assign the same transversal weight to all the subspaces with the same dimension. We also need to find the size and the distribution of elements in $\mathcal{B}_{P,r}(x)$. The general formula is given in [4] but we only study the case $r = 1$. Given x with dimension k in X_P, there are $\begin{bmatrix} k \\ k-1 \end{bmatrix}_2 = 2^k - 1$ subspaces of dimension $k-1$ in $\mathcal{B}_{P,1}(x)$, where

$$\begin{bmatrix} n \\ m \end{bmatrix}_2 = \frac{(2^n-1)(2^{n-1}-1)\cdots(2^{n-m+1})}{(2^m-1)(2^{m-1}-1)\cdots(2^1-1)}$$

is the number of subspaces of dimension m in a space of dimension n. There are also $\frac{2^n-2^k}{2^k} = 2^{n-k} - 1$ subspaces of dimension $k+1$ in $\mathcal{B}_{P,1}(x)$ that include x. Therefore, there are $(2^k - 1) + (2^{n-k} - 1) + 1$ elements in $\mathcal{B}_{P,1}(x)$. So,

$$\tau^*(\mathcal{H}(\mathcal{G}_P, r)) = \min\left\{\sum_{k=0}^{n} w_k \begin{bmatrix} n \\ k \end{bmatrix}_2 : \forall\, 0 \leqslant k \leqslant n \right. \quad (8)$$

$$\left. w_k + (2^k-1)w_{k-1} + (2^{n-k}-1)w_{k+1} \geqslant 1, w_k \geqslant 0\right\}.$$

It is shown that there exist automorphisms which map a fixed subspace of dimension k to a fixed subspace of dimension $n-k$ (see [3].) So, subspaces of dimension k and $n-k$ are also in the same equivalence classes and we assign the same weights to them. Also note that $\begin{bmatrix} n \\ k \end{bmatrix}_2 = \begin{bmatrix} n \\ n-k \end{bmatrix}_2$. Hence we benefit from a very nice symmetry and we set $w_k = w_{n-k}$ to halve both the number of constraints and the parameters in the linear programming. The optimal transversal weights we found for $n \leqslant 11$ are listed in Table III.

It is interesting to see that $w_{\lfloor \frac{n}{2} \rfloor} = 0$ for all $n > 2$, which is no surprising since $\begin{bmatrix} n \\ \lfloor \frac{n}{2} \rfloor \end{bmatrix}_2$ is the largest coefficient in the cost function. This leads us to a greedy approach of starting from the middle, which has the highest impact on cost function; minimizing it, i.e. $w_{\lfloor \frac{n}{2} \rfloor} = 0$; and then moving toward the tails where we pick the least possible value to satisfy the constraints. We call it the ***greedy*** weight assignment, which is expressed as

TABLE III
Projective codes: upper bounds and weights for $r = 1$

n	$w_0^*, w_1^*, \cdots, w_{\lfloor \frac{n}{2} \rfloor}^*$	GSPB	[1]
2	1, 0	1	-
3	1, 0	2	-
4	0.83, 0.17, 0	6	6
5	0.67, 0.34, 0	22	20
6	0, 0.30, 0.07, 0	132	124
7	0, 0.29, 0.15, 0	834	776
8	1, 0, 0.14, 0.03, 0	9460	9268
9	1, 0, 0.13, 0.07, 0	116656	107419
10	1, 0, 0, 0.066, 0.016, 0	2566390	-
11	1, 0, 0, 0.065, 0.032, 0	62462160	-

$$\begin{aligned} &w^*_{\lceil \frac{n}{2} \rceil} = 0, \quad \text{and for all } k\text{:} \quad 0 \leqslant k < \lfloor \frac{n}{2} \rfloor, \\ &w_k^* = \max\{\frac{1 - w_{k+1}^* - (2^{n-k-1}-1)w_{k+2}^*}{2^{k+1}-1}, 0\}, \qquad (9) \\ &w_k^* = w_{n-k}^*, \text{ and if } w_0^* = w_1^* = 0, \text{ then } w_0^* = 1. \end{aligned}$$

It is clear that the greedy output has the transversal property and lies in the feasible set. In fact, w^* for $k < \lfloor \frac{n}{2} \rfloor$ is given by

$$w_k^* = \begin{cases} \frac{1}{2^{k+1}-1} & \text{if } k \equiv \lfloor n/2 \rfloor - 1 \bmod 4 \\ \frac{2}{2^{k+2}-1} & \text{if } k \equiv \lfloor n/2 \rfloor - 2 \bmod 4 \\ 0 & \text{otherwise,} \end{cases}$$

with the only exception of

$$w_k^* = \begin{cases} \frac{1}{2(2^{k+1}-1)} & \text{if } k = \frac{n}{2} - 1 \\ \frac{2^{k+3}-3}{(2^{k+1}-1)(2^{k+2}-2)} & \text{if } k = \frac{n}{2} - 2 \end{cases}$$

for n even. The following theorem also shows the optimality of the greedy assignment in our scheme (See [5] for the proof.)

Theorem 9. *Let $\mathcal{G}_P$ be the associated graph with projective code when $\mathbb{F}_q^n$ is the space and w^* be as defined in (9), then*

$$\tau^*(\mathcal{H}(\mathcal{G}_P, r)) = \sum_{k=0}^{n} w_k^* \begin{bmatrix} n \\ k \end{bmatrix}_2.$$

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